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Impulsive Optimal Control of Hybrid Finite-Dimensional Lagrangian Systems

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Abstract

The scope of this dissertation addresses numerical and theoretical issues in the impulsive control of hybrid finite-dimensional Lagrangian systems. In order to treat these aspects, a modeling framework is presented based on the measure-differential inclusion representation of the Lagrangian dynamics. The main advantage of this representation is that it enables the incorporation of set-valued force laws and control laws on acceleration and velocity level concisely. This property of the measure-differential inclusion representation renders the description of the hybrid behaviour of Lagrangian systems in the framework of set-valued control and force laws possible. Based on the MDI representation of Lagrangian dynamics the impactive blocking is analysed as a set-valued impulsive unbounded control law. The application to mechanical systems with impulsively-blockable degrees of freedom is presented. The numerical application of this set-valued control law is the formulation of a nonlinear programming (NLP) algorithm for underactuated mechanical systems with impulsively-blockable DOF. The natural numerical treatment of the measure-differential inclusion representation is based on Moreau's sweeping process. By applying this discretisation scheme together with an augmented Lagrangian based NLP method that performs the minimisations with a modified conjugate gradients method an optimisation scheme is presented. A numerical example is applied to the impulsive optimal control of a manipulator with one impulsively-blockable degrees of freedom. A further numerical method is introduced for the class of switching Lagrangian systems. This numerical method is a shooting method that performs the numerical integrations based on the sweeping process and the minimisations by making use of the augmented Lagrangian concept. The augmented Lagrangian is minimised by an optimisation method that relies on function value comparisons. This relatively easily implemented numerical method is applied to a wheeled robot which is a tenth-order dynamical system. This system has four different operating modes and time and control effort (quasi-) optimal trajectories are presented. The theoretical results of the dissertation include the statement and the derivation of necessary conditions for the impulsive optimal control of finite-dimensional Lagrangian systems. In this analysis, Lagrangian systems are considered on which the impulses are induced solely by the impulsive control action. The challenge in the derivation of these necessary conditions has been the concurrent discontinuity of state and costate on a Lebesgue negligible time instant. In order to tackle this problem, the instances of impulsive control action are considered as an internal boundaries on the time domain. By the introduction of the concepts of internal boundary variations and discontinuous transversality conditions by the author this problem is resolved and necessary conditions for mechanical systems in the first-order and second-order representations are derived. The discontinuous transversality conditions that result from the consideration of the internal boundary variations in the time domain are discovered and analysed by the author of the dissertation and are applied to the impulsive optimal control of Lagrangian systems.

Zusammenfassung

Im Rahmen dieser Dissertation wurden numerische und theoretische Problemstellungen im Bereich der impulsiven Steuerung hybrider endlich-dimensionaler Lagrangescher Systeme behandelt. In der Behandlung der Problemstellungen wurde die dynamische Modellierung auf das mathematische Konzept der Massdifferentialinklusionen (MDI) basiert. Das Konzept der Massdifferentialinklusion ermöglicht die einwandfreie und präzise Eingliederung von mengenwertigen Steuer-, und Kraftgesetzen im Modell. Diese Eigenschaft der MDI Darstellung ist der Hauptvorteil und ermöglicht die Beschreibung des hybriden Verhaltens Lagrangescher Systeme mit Hilfe von mengenwertigen Kraft und Steuergesetzen. Basierend auf die MDI Darstellung der Lagrangeschen Dynamik wird die stosshafte Blockierung analysiert und als ein mengenwertiges unbegrenztes impulsives Steuergesetz eingeführt. Die Anwendung dieses Steuergesetzes auf mechanische Systeme mit stosshaft blockierbaren Freiheitsgraden wird präsentiert. Als numerische Anwendung des mengenwertigen Steuergesetzes in der impulsiven optimalen Steuerung wird ein nichtlinearer Programmierungsalgorithmus (NLP) beschrieben, der die Bestimmung optimaler Trajektorien für unteraktuierte Lagrangesche Systeme mit stosshaft blockierbaren Gelenken bezweckt. Die natürliche numerische Behandlung der Massdifferentialinklusionen basiert auf dem Sweeping Algorithmus von Moreau. Der NLP-Algorithmus vereinigt in sich das Diskretisierungsschema des Sweeping Algorithmuses und die erweiterte Lagrangesche Methode in Optimierung. Die Minimierung der erweiterten Lagrangeschen Funktion wird mit einem modifizierten konjugierten Gradientenverfahren bewältigt. Ein numerisches Beispiel wird angeführt, das die numerische Ermittlung der impulsiven optimalen Steuerung eines Manipulators mit einem stosshaft blockierbaren Gelenk beschreibt. Im weiteren wird eine numerische Methode vorgestellt, das ein Schiessverfahren ist, in der die numerische Integration mit dem Sweeping Algorithmus ausgeführt wird. Dieses Schiessverfahren ermöglicht die Bestimmung optimaler Trajektorien von einer bestimmten Klasse von Lagrangeschen Systemen, die im Rahmen dieser Arbeit umschaltende Lagrangesche Systeme genannt werden. Die zugehörige erweiterte Lagrangesche Funktion wird mittels eines Optimierungsverfahrens minimiert, das auf Funktionswertvergleich beruht. Die im Vergleich zu NLP Methoden relativ einfach implementierbare Methode wird zur optimalen Trajektorienbestimmung eines Roboters mit Rädern angewendet, der als ein dynamisches System zehnter Ordnung beschrieben wird. Dieses System hat vier Moden, die sich voneinander aufgrund des Haft-Gleit Zustandes der Räder unterscheiden. Zeit und steuerungsaufwandsoptimierte Trajektorien werden präsentiert.

Der theoretische Beitrag der Arbeit ist die Bestimmung der notwendigen Bedingungen für impulsive optimale Steuerungen endlich-dimensionaler Lagrangescher Systeme. In der Analyse wird angenommen, dass die impulsive Steuerung die einzige stosshafte Anregung des Lagrangeschen Systems darstellt. Die Herausforderung in der Herleitung der notwendigen Bedingungen ist die Behandlung der gleichzeitigen Unstetigkeit in den Geschwindigkeiten des mecha-

nischen Systems und der adjungierten Zuständen. Um diese Herausforderung zu meistern wurden die Momente der impulsiven Steuerungseingriffe als innere Grenzen im Zeitbereich betrachtet. Der Verfasser dieser Dissertation führt deswegen die Variationen an den inneren Rändern im Zeitbereich zusammen mit den diskontinuierlichen Transversalitätsbedingungen ein, um diese Komplikation zu beheben. Auf diesen diskontinuierlichen Transversalitätsbedingungen basierend wurden notwendige Bedingungen in erster und zweiter Ordnungsdarstellung hergeleitet, die von einer optimalen Bahn erfüllt werden müssen. Die aus den inneren Randvariationen hervorgehenden diskontinuierlichen Transversalitätsbedingungen wurden vom Verfasser dieser Dissertation in eigenständiger Arbeit erfunden, analysiert und auf impulsive Steuerung Lagrangescher Systeme angewendet.

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Contents

1	Introduction	1
1.1	Modeling	3
1.2	Optimal Control and Hybrid Systems	5
1.3	Discontinuous Transversality Conditions	8
1.4	Numerical Analysis	10
2	Modeling of Hybrid FDLS	19
2.1	Preliminaries	19
2.2	Lagrange Equations in Impulsive Control Form	23
2.3	The MDI of Motion for FDLS with Controls	25
2.3.1	LCP Representation of MDI	27
2.3.2	Impact Equation	31
2.4	Underactuated Manipulators with Impactive Blocking	35
2.5	Complementarity Description of Impulsive Blocking	36
2.5.1	A Case Study	43
2.6	Impactive Underactuated Manipulators	45
2.6.1	Post-transition State and Discontinuity	45
2.6.2	Lagrangian Dynamics in Different Phases of Motion	48
2.6.3	Change in Mechanical Energy and Impuls and Dissipation	50
2.6.4	Example: Planar Double Pendulum with One Blockable DOF	51
3	Numerical Methods for FDLS	55
3.1	Preliminaries	58
3.1.1	Complementarity Problems and Variational Inequalities	67
3.2	Properties of the Augmented Lagrangian Method	72
3.2.1	The Lagrangian and the Augmented Lagrangian	72
3.2.2	Normality and Degeneracy of the NLP Problem	75
3.2.3	Local Analysis of the Augmented Lagrangian Function	78
3.2.4	Global Analysis of the Augmented Lagrangian Function	81

3.2.5	Comparison of Strategies	84
3.3	NLP Method for MPEC	86
3.3.1	The Transcription of a mechanical MDI Optimal Control Problem into a NLP	89
3.3.2	Formulas for the Determination of the Numerical Gradient of the Aug- mented Lagrangian	93
3.3.3	The NLP Algorithm	95
3.3.4	Implementation of the Algorithm	99
3.4	Numerical Example	99
3.4.1	Case A	103
3.4.2	Case B	110
3.4.3	Case C	116
3.5	Augmented Lagrangian based Shooting Method	124
3.6	Contact Dynamics and Integration	126
3.6.1	Formulation of Shooting Method	129
3.7	Model of the Diff-Drive Robot	131
3.7.1	Determination of Normal Contact Force Differential Measures	136
3.8	Numerical Results	139
3.8.1	Case A	140
3.8.2	Case B	140
3.8.3	Case C	141
3.8.4	Case D	148
3.8.5	Simplectic and Dissipative Properties of the Sweeping Process	150
3.8.6	Convergence Behaviour of Case B	153
4	Discontinuous Transversality and Impulsive Control	157
4.1	Introduction	157
4.2	Preliminaries	159
4.2.1	Regularity of Sets	162
4.3	Impulsive Generalised Problem of Bolza	164
4.4	Statement of the Optimal Control Problem	166
4.5	Discontinuous Transversality Conditions	171
4.5.1	Internal Boundary Variations	172
4.5.2	Discontinuous Transversality Conditions	173
4.5.3	Total Directional Derivatives of The Value Function J	179
4.6	Necessary Conditions	180
4.7	Necessary Conditions for Impactive Underactuated Systems	183

5	Hamiltonian Approach	185
5.1	Derivation of First-order Conditions	185
5.2	Generalised Problem of Bolza	186
5.3	Statement of IOC	189
5.4	Necessary Conditons	197
5.5	Case Study	199
6	Discussion and Conclusion	203
6.1	Results on Numerical Methods	203
6.2	Variational Results	208
	Bibliography	211
	Curriculum Vitae	222

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List of Symbols

$\succcurlyeq$	Elementwise greater equal than
$\preccurlyeq$	Elementwise less equal than
$ \cdot $	Absolute value of the argument
$\nabla_{\mathbf{x}} \mathbf{y}$	Gradient of $\mathbf{y}$ w.r.t $\mathbf{x}$
$\mathcal{A} \subset \mathcal{B}$	$\mathcal{A}$ is subset of $\mathcal{B}$
$\mathcal{A} \supset \mathcal{B}$	$\mathcal{A}$ is superset of $\mathcal{B}$
$N(\mathcal{B})$	Number of elements of set $\mathcal{B}$
$\langle \cdot, \cdot \rangle$	scalar product or dual pairing of the arguments
$\mathbf{a} \times \mathbf{b}$	cross product of vectors $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a} \perp \mathbf{b}$	Scalar product of $\mathbf{a}$ and $\mathbf{b}$ is zero
$\mathbb{B}$	Open unit ball in Euclidean space
$ \mathbf{x} $	Euclidean norm of $\mathbf{x}$
$d_{\mathcal{C}}(\mathbf{x})$	Euclidean distance of $\mathbf{x}$ from $\mathcal{C}$
$\text{int } \mathcal{C}$	Interior of $\mathcal{C}$
$\text{bdy } \mathcal{C}$	Boundary of $\mathcal{C}$
$\bar{\mathcal{C}}$	Closure of $\mathcal{C}$
$\mathcal{N}_{\mathcal{C}}^{\text{P}}(\mathbf{x})$	Proximal normal cone to $\mathcal{C}$ at $\mathbf{x}$
$\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x})$	Strict normal cone to $\mathcal{C}$ at $\mathbf{x}$
$\mathcal{N}_{\mathcal{C}}(\mathbf{x})$	Limiting normal cone to $\mathcal{C}$ at $\mathbf{x}$
${}^{\infty}\mathcal{N}_{\mathcal{C}}^{\text{P}}(\mathbf{x})$	Singular proximal normal cone to $\mathcal{C}$ at $\mathbf{x}$
${}^{\infty}\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x})$	Singular strict normal cone to $\mathcal{C}$ at $\mathbf{x}$
${}^{\infty}\mathcal{N}_{\mathcal{C}}(\mathbf{x})$	Singular limiting normal cone to $\mathcal{C}$ at $\mathbf{x}$
$\mathcal{T}_{\mathcal{C}}(\mathbf{x})$	Bouligand tangent cone to $\mathcal{C}$ at $\mathbf{x}$
$\bar{\mathcal{T}}_{\mathcal{C}}(\mathbf{x})$	Clarke tangent cone to $\mathcal{C}$ at $\mathbf{x}$
$\text{epi } f$	Epigraph of f
$\partial^{\text{P}} f(\mathbf{x})$	Proximal subdifferential of f at $\mathbf{x}$
$\hat{\partial} f(\mathbf{x})$	Strict subdifferential of f at $\mathbf{x}$
$\partial f(\mathbf{x})$	Limiting subdifferential of f at $\mathbf{x}$
$\partial_P^{\infty} f(\mathbf{x})$	Asymptotic proximal subdifferential of f at $\mathbf{x}$
$\hat{\partial}_P^{\infty} f(\mathbf{x})$	Asymptotic strict subdifferential of f at $\mathbf{x}$
$\partial^{\infty} f(\mathbf{x})$	Asymptotic limiting subdifferential of f at $\mathbf{x}$

$\text{dom } f$	(Effective) domain of f
$\text{Gr } \mathcal{F}$	Graph of $\mathcal{F}$
$\text{epi } f$	Epigraph of f
$f^0(\mathbf{x}; \mathbf{v})$	Generalised directional derivative of f at $\mathbf{x}$ in the direction of $\mathbf{v}$
$\Psi_{\mathcal{C}}(\mathbf{x})$	Indicator function of the set $\mathcal{C}$ at the point $\mathbf{x}$
$\nabla f(\mathbf{x})$	Gradient vector of f at $\mathbf{x}$
$\mathbf{x}_i \xrightarrow{\mathcal{C}} \mathbf{x}$	$\mathbf{x}_i \rightarrow \mathbf{x}$ and $\mathbf{x}_i \in \mathcal{C}$, $\forall i$
$\mathbf{x}_i \xrightarrow{f} \mathbf{x}$	$\mathbf{x}_i \rightarrow \mathbf{x}$ and $f(\mathbf{x}_i) \rightarrow f(\mathbf{x})$, $\forall i$
$\text{supp } \mu$	Support of the measure μ

List of Abbreviations

$\mathcal{BV}(\mathcal{I}; \mathbb{R}^n)$	Bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{LCBV}(\mathcal{I}; \mathbb{R}^n)$	Left-continuous bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{RCBV}(\mathcal{I}; \mathbb{R}^n)$	Right-continuous bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{LBV}(\mathcal{I}; \mathbb{R}^n)$	Locally bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{LCLBV}(\mathcal{I}; \mathbb{R}^n)$	Left-continuous locally bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{RCLBV}(\mathcal{I}; \mathbb{R}^n)$	Right-continuous locally bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{AC}(\mathcal{I}; \mathbb{R}^n)$	Absolutely continuous functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
LCP	Linear Complementarity Problem
NCP	Nonlinear Complementarity Problem
MCP	Mixed Complementarity Problem
MPEC	Mathematical Programs with Complementarity Constraints
NLP	Nonlinear Programming
TOHLS	Trajectory Optimisation of Hybrid Lagrangian Systems
FDLS	Finite-Dimensional Lagrangian System
UPR($\cdot$)	Unilateral primitive
SGN($\cdot$)	Set-valued Signum Relation

List of Tables

3.1	System parameters of the manipulator for cases A, B, C.	101
3.2	Correspondence between inequalities and dual multipliers in the NLP of the planar double pendulum.	101
3.3	Case A: The mode sequence, duration and intervals of modes.	105
3.4	Case A: Optimisation parameters, initial and final states of the optimisation. . .	105
3.5	Case B: The mode sequence, duration and intervals of modes.	110
3.6	Case B: Optimisation parameters, initial and final states of the optimisation. . .	111
3.7	Case C: The mode sequence, duration and intervals of modes.	117
3.8	Case C: Optimisation parameters, initial and final states of the optimisation. . .	117
3.9	Notation Convention.	132
3.10	The classification of modes by making use of the relative contact state of the rear wheels (The modes that emanate due to the stick-slide situations of the frontal contact C_F are neglected, $\mu_f \approx 0$)	134
3.11	Numerical values of physical parameters of the robot. The numerical values are obtained from the CAD of a real mobile robot.	137
3.12	Numerical parameters of the optimisations in Maneuvers A, B, C and D. Here c_t and c_d are penalties imposed on constraints on end time and on the deviation from the final state, respectively.	140

List of Figures

2.1	Unilateral primitive.	21
2.2	Decomposition of the Signum relation into two Uprs.	21
2.3	General contact kinematics.	26
2.4	Decomposition of the set-valued Sgn relation into two Unilateral Primitives. . .	30
2.5	(a) A perfect bilateral constraint is equivalent to a normal control force that takes infinite values, (b) For values of N less than infinity and greater zero, the blocking action has friction characteristics, (c) if $n = 0$ there is no braking force.	38
2.6	The set-valued signum relation and its decomposition into two unilateral primitives that represents relation between the discretised differential measure of normal blocking force to the discretised differential measure of the control force.	39
2.7	(a) If $\hat{N}$ rises to infinity, the relative velocity in the next moment drops to zero, (b) For values of $\hat{N}$ less than infinity and greater zero, the blocking action has friction characteristics with respect to the relative velocity, (c) if $\hat{N} = 0$ there is no blocking in the next moment.	40
2.8	The unilateral primitive about the relation of the blocking normal force to the relative acceleration $\dot{\gamma}_b^+$	40
2.9	The UPR about the relation of the differential measure of blocking normal impulsive force to the relative velocity $\gamma_b^+ \in \mathcal{RCLBV}$	40
2.10	The decomposition of the Upr relation in figure 2.9 into two unilateral primitives.	41
2.11	The decomposition of the Upr relation in figure 2.8 into two unilateral primitives.	41
2.12	The 2-DOF linear mechanism	43
2.13	Planar double pendulum with one impactively blockable degrees of freedom. . .	51
3.1	A nonconvex function f , its epigraph, and the Generalised Subdifferential at $\mathbf{x}_0$.	62
3.2	The functions Φ_{FB} and $-\Phi_{FB}$	69
3.3	The limiting subdifferentials of Φ_{FB} and $-\Phi_{FB}$ at the origin.	70
3.4	The decomposition of the Upr relation into two unilateral primitives in discretised form.	87

3.5	The set-valued signum relation and its decomposition into two unilateral primitives that represents relation between the discretised differential measure of normal blocking force to the discretised differential measure of the control force in discretised form.	88
3.6	The simplified flowchart of the algorithm.	97
3.7	The flowchart of the conjugate gradients minimisation.	98
3.8	The parameters of the planar double pendulum.	100
3.9	Case A: The optimal evolution of the generalised velocities $\dot{\alpha}, \dot{\beta}$, relative velocity $\dot{\alpha} - \dot{\beta}$, generalize positions α, β and relative position $\alpha - \beta$. (red lines mark the transition times)	104
3.10	Case A: Maneuver of the double pendulum (Color code: For link 2 white is blocked, black is unblocked, for link 1 green position at transition interval). . . .	105
3.11	Case A: The optimal evolution of the discretised differential measures N, Γ , the torque τ and the slack velocity γ_{br} (red lines mark the transition times).	106
3.12	Case A: The numerical residue of the discretised equations of motion approx. by $\frac{\mathbf{B}[k]}{\Delta t}$ (red lines mark the transition times).	106
3.13	Case A: Residues of the reformulation functions $\Phi_R, \Phi_L, \Phi_{NR}$ and Φ_{NL} (red lines mark the transition times).	107
3.14	Case A: The optimal evolution of the dual multipliers ${}^mL_\alpha, {}^mL_\beta, {}^pL_\alpha$ and ${}^pL_\beta$ (red lines mark the transition times).	107
3.15	Case A: The time evolution of the costate dynamics in the second-order form (red lines mark the transition times).	108
3.16	Case A: The optimal evolution of the dual multipliers ${}^mL_R, {}^mL_L, {}^pL_R$ and pL_L (red lines mark the transition times).	108
3.17	Case A: The optimal evolution of the dual multipliers ${}^mL_{NR}, {}^mL_{NL}, {}^pL_{NR}$ and ${}^pL_{NL}$ (red lines mark the transition times).	109
3.18	Case A: The residues of the complementarities (red lines mark the transition times).	109
3.19	Case B: The optimal evolution of the generalised velocities $\dot{\alpha}, \dot{\beta}$, relative velocity $\dot{\alpha} - \dot{\beta}$, generalised positions α, β and relative position $\alpha - \beta$. red lines mark the transition times)	111
3.20	Case B: Maneuver of the double pendulum (Color code: For link 2 white is blocked, black is unblocked, for link 1 green position at transition interval). . . .	112
3.21	Case B: The optimal evolution of the discretised differential measures N, Γ , the torque τ and the slack velocity γ_{br} (red lines mark the transition times).	112
3.22	Case B: The numerical residue of the discretised equations of motion of motion approx. by $\frac{\mathbf{B}[k]}{\Delta t}$ (red lines mark the transition times).	113

3.23	Case B: Residues of the reformulation functions Φ_R , Φ_L , Φ_{NR} and Φ_{NL} (red lines mark the transition times).	113
3.24	Case B: The optimal evolution of the dual multipliers ${}^m_B L_\alpha$, ${}^m_B L_\beta$, ${}^p_B L_\alpha$ and ${}^p_B L_\beta$ (red lines mark the transition times).	114
3.25	Case B: The time evolution of the costate dynamics in the second-order form (red lines mark the transition times).	114
3.26	Case B: The optimal evolution of the dual multipliers ${}^m L_R$, ${}^m L_L$, ${}^p L_R$ and ${}^p L_L$ (red lines mark the transition times).	115
3.27	Case B: The optimal evolution of the dual multipliers ${}^m L_{NR}$, ${}^m L_{NL}$, ${}^p L_{NR}$ and ${}^p L_{NL}$ (red lines mark the transition times).	115
3.28	Case B: The residues of the complementarities (red lines mark the transition times).	116
3.29	Case C: The optimal evolution of the generalised velocities $\dot{\alpha}$, $\dot{\beta}$, relative velocity $\dot{\alpha} - \dot{\beta}$, generalize positions α , β and relative position $\alpha - \beta$. red lines mark the transition times)	118
3.30	Case C: Maneuver of the double pendulum (Color code: For link 2 white is blocked, black is unblocked, for link 1 green position at transition interval).	119
3.31	Case C: The optimal evolution of the discretised differential measures N , Γ , the torque τ and the slack velocity γ_{br} (red lines mark the transition times).	119
3.32	Case C: The numerical residue of the discretised equations of motion approx. by $\frac{\mathbf{B}[k]}{\Delta t}$ (red lines mark the transition times).	120
3.33	Case C: Residues of the reformulation functions Φ_R , Φ_L , Φ_{NR} and Φ_{NL} (red lines mark the transition times).	120
3.34	Case C: The optimal evolution of the dual multipliers ${}^m_B L_\alpha$, ${}^m_B L_\beta$, ${}^p_B L_\alpha$ and ${}^p_B L_\beta$ (red lines mark the transition times).	121
3.35	Case C: The time evolution of the costate dynamics in the second-order form (red lines mark the transition times).	121
3.36	Case C: The optimal evolution of the dual multipliers ${}^m L_R$, ${}^m L_L$, ${}^p L_R$ and ${}^p L_L$ (red lines mark the transition times).	122
3.37	Case C: The optimal evolution of the dual multipliers ${}^m L_{NR}$, ${}^m L_{NL}$, ${}^p L_{NR}$ and ${}^p L_{NL}$ (red lines mark the transition times).	122
3.38	Case C: The residues of the complementarities (red lines mark the transition times).	123
3.39	Contact forces and motor moments on the simplified model.	132
3.40	The degrees of freedom and coordinate frames of the wheeled robot.	132
3.41	Case A: Contact forces and contact relative velocities.	141
3.42	Case B: Contact forces and contact relative velocities.	142

3.43	Case A: Number of DOF during the maneuver.	142
3.44	Case B: Number of DOF during the maneuver.	142
3.45	Case A: The final time t_f during successive minimisations.	143
3.46	Case B: The final time t_f during successive minimisations.	143
3.47	Case A: The evolution of the trajectory of the CM.	143
3.48	Case B: The evolution of the trajectory of the CM.	143
3.49	Case C: Number of DOF during the maneuver.	143
3.50	Case D: Number of DOF during the maneuver.	143
3.51	Case C: Control Moments and phases of chassis DOF.	144
3.52	Case D: Control Moments and phases of chassis DOF.	144
3.53	Case C: Final time t_f	145
3.54	Case D: Final time t_f	145
3.55	Case C: The control effort.	145
3.56	Case D: The control effort.	145
3.57	Case C: Contact forces and contact relative velocities.	145
3.58	Case D: Contact forces and contact relative velocities.	146
3.59	Case C: Contact forces and contact relative velocities.	146
3.60	Case A: Control Moments and phases of chassis DOF.	147
3.61	Case B: Control Moments and phases of chassis DOF.	147
3.62	ΔT_{err}^j for Maneuver A.	149
3.63	ΔT_{err}^j for Maneuver B.	149
3.64	ΔT_{err}^j for Maneuver C.	149
3.65	ΔT_{err}^j for Maneuver D.	149
3.66	$\frac{W_C}{W_D}$ for Maneuvers in intermediate Optimisation stages.	149
3.67	$W_C + W_D$ for Maneuvers in intermediate Optimisation stages.	149
3.68	Case A: Errors in Energy.	150
3.69	Case B: Errors in Energy.	150
3.70	Case C: Errors in Energy.	150
3.71	Case D: Errors in Energy.	150
3.72	Case B: Evolution and Convergence Behaviour of x and $\dot{x}$ trajectories.	153
3.73	Case B: Evolution and Convergence Behaviour of y and $\dot{y}$ trajectories.	154
3.74	Case B: Evolution and Convergence Behaviour of ϕ and $\dot{\phi}$ trajectories.	154
3.75	Case B: Evolution of Ψ_R , Ψ_L , $\dot{\Psi}_R$ and $\dot{\Psi}_L$ trajectories.	155
3.76	Case B: Evolution and Convergence Behaviour of M_L and M_R trajectories.	155
3.77	Case B: Evolutions of DOF x, y and ϕ as phase diagrams.	156

Chapter 1

Introduction

Many classes of dynamical systems are exhibiting at certain intervals in their evolution rapid changes. In modeling such systems, it is sometimes more reasonable and convenient to neglect the duration of these sudden and highly dynamic alterations and assume that the state jumps. In doing so, it is then necessary to classify the character of those jumps. One classification relies on whether the jump is intrinsic to the system or is imposed externally on the system. Further, the conditions under which such jumps may occur is also important. If very high control forces are applied on a dynamical system, then the system may exhibit such jumps, which can be seen as an external action applied to the system to change its course of evolution drastically. In this work, I aimed at investigating the impulsive optimal control of hybrid finite-dimensional Lagrangian systems from this view of angle. In the realm of dynamical systems, Lagrangian systems are members of the royal family. Lagrangian systems comprise the major part of the physical reality in which we are existing. Even, without looking very closely, one immediately sees, that they exhibit weird nonsmooth and discontinuous phenomena. Discontinuous, nonsmooth dynamics encompasses the physical environment, we exist, yet it took so long to understand them. Recent decades of research witnessed the advent of nonsmooth analysis, that also pushed our understanding of nonsmooth Lagrangian systems forward, by providing our scientific zeal the required means to disclose its "secrets". In this thesis a unified framework is developed for the determination of optimal trajectories of structure-variant finite-dimensional Lagrangian systems. This thesis aims at providing a profound analysis for the above aspects of hybrid mechanical trajectory optimisation as well as proposing numerical methods for the determination of the trajectories. One of the main contributions of this work is the development a unified framework of modeling and optimisation that enables the determination of the sequence of modes and a sequence of transitions as an outcome of the optimisation without prespecifying them before. The class of finite-dimensional Lagrangian systems include the important class of rigid-body multi-body systems. The underlying Lagrangian structure, enables the consideration of wider classes of physical systems. In general, in the method

of finite-elements (FEM) the infinite-dimensional problems of the continua are converted to finite-dimensional Lagrangian setting. After application of the Ritz-Galerkin Ansatz problems in continuum mechanics become also finite-dimensional, amenable to the techniques that are discussed here. However, the focus in this thesis are multi-body dynamical systems. This type of problems arise in the trajectory optimisation of hybrid mechanical systems regularly. Possible areas of application involve legged locomotion, manipulators with blockable degrees of freedom (DOF), aerospace applications where system parameters such as inertial parameters change discontinuously, robotic applications that involve contacts such as grasping.

The determination of the optimal trajectory of hybrid mechanical systems is expected to deal with several aspects satisfactorily. These aspects are classified in seven broad classes as

- Optimality
- Reachability
- Determination of a feasible sequence of modes
- Optimality of the transitions
- Index Set Management
- Nonlinear Dynamics
- Nonsmoothness

Optimality is related to the determination of necessary and sufficient conditions. The necessary conditions for impulsive optimal control of finite-dimensional Lagrangian systems is stated and derived by making use of nonsmooth analysis. The optimality of a hybrid trajectory requires the assessment of the optimality of the transitions. The transitions between modes of mechanical systems are accompanied by discontinuities on acceleration and/or velocity level. As the needs of optimal control enabled the derivation of new methods in variational analysis, the derivation of necessary conditions for Lagrangian systems for impulsive control cases requires the proposition of several new concepts as well.

Reachability is the characterisation under which conditions a final state is attained and assessment whether under given constraints a desired final state is reachable. In the convex case, there are necessary conditions that enable the statement of those conditions. Numerically, reachability is characterised by the conditions under which a solution exists and converges.

A hybrid system approach to structure-variant mechanical systems necessitates the determination of a sequence of modes, which includes determining the order of succession and duration of each mode. A mode of a hybrid mechanical process, is characterised by a parameter dependent set of differential equations. The modes of a mechanical process may differ from each

other on the basis of the number of differential equations or on the basis of parameters. The modes of a hybrid mechanical process, may consist of a infinite set, i.e. depending on the value system parameters such as inertias and geometric dimensions.

Index set management is the task to manage the transition conditions under which during a hybrid mechanical process a system may change modes. The change of modes may be triggered by the control strategy or by some autonomous action such as stick-slip transition in the system.

The inherent property of Lagrangian systems that they are nonlinear, poses problems in the analysis due to the induced nonconvexity in the infinite and finite-dimensional analysis of the problem.

Nonsmoothness is a property that is encountered in the finite and infinite dimensional forms of the problem due to several reasons. The main source of nonsmoothness is the discontinuity of the generalised velocity of the system due to impulsive force interactions. In the modeling, set-valued force and control elements are needed to characterise the systems which are nonsmooth. Numerically, the theory of nonsmooth analysis is needed to deal with such problems. In the derivation of the infinite-dimensional necessary conditions and the minimisation of the generalised Bolza functional nonsmooth analysis techniques are used.

1.1 Modeling

There has been much interest in the research of modeling discontinuities and nonlinearities in multibody systems. A compact overview is provided in the book written by Brogliato [19]. An optimisation approach can in the opinion of the author not succeed if the nature of the system considered is not analysed profoundly. Already at times of Newton the issue of impact and the discontinuity of the velocity has been a research object. As also observed by Newton, an impact in mechanics is defined as a discontinuity in the generalised velocities of a mechanical system, which is induced by impulsive forces. Impulsive forces, on the other hand, are defined in the distributional sense. The mathematical framework, is that impulsive forces are represented by Dirac distributions. In [72] Orlov provides a good overview on the application of Schwartz' distributions theory in nonlinear setting. The discontinuities arising from impacts and stick-slip transitions are primarily contact phenomena, which concur temporally and spatially. The spatial concurrence of discontinuity is due to the fact that discontinuities on velocity level (e.g. collisions) can occur along with discontinuities on acceleration level (e.g. stick-slip transitions). In recent years, several works have been presented in order to establish the relations between complementarity dynamical systems and hybrid systems. There are general results in literature that investigate the relation between different representations of dynamical systems and hybrid systems as published in works such as Brogliato *et al.* [21], and Heemels *et al.* [47]. Recent research showed that such rigid-body systems can best be described by variational inequalities

which lead to nonlinear and linear complementarity type of systems to be solved in order to obtain the accelerations/velocities and forces. The properties of the optimal control problem derive from the underlying modeling approach. Optimal control of hybrid systems is addressed in several publications by Benghea *et al.* [11], Borelli *et al.* [16], Potcnik [75] and Shaik *et al.* [93] on various field of applications, in which the modeling is based on approaches to the ones similar as in the works of Bemporad *et al.* [10] and Branicky *et al.* [17]. The treatment of discontinuous transitions and the combinatorial nature of mode sequencing are partially treated in these publications about optimal control, due to the modeling approach chosen. In [117] by Yunt *et al.*, the measure-differential inclusion (MDI) based modeling of mechanical hybrid systems is proposed and the suitability from the viewpoints above presented. In the modeling considered in this work, impulsive forces can arise autonomously, due to effects such as collisions or controlled/nonautonomously, due to actions such as blocking some DOF. The hybrid optimal control requires the consideration of an uncommon concept of control, namely, controls of unbounded, impulsive and set-valued type. The existence of force and impulsive/discrete type of controls influences through the solution of the complementarity problem the course of system trajectories. The presence of impulsive forces require to solve impact equations and constitutive laws that relate post- and pre-impact velocities of the system. The topic of impact with and without friction is investigated in reference by Glocker [43], profoundly. In the modeling considered in this work, impulsive forces can arise autonomously, due to effects such as collisions or controlled/nonautonomously, due to actions as blocking of manipulator degrees of freedom suddenly. The introduced framework has the ability to model control of hybrid mechanical systems with discontinuous transitions among different system modes. The MDI presentation provides a strong tool to describe structure-variant systems with explicit or implicit phase transitions. The advantages to represent hybrid Lagrangian systems as MDI's and treat them with related numerical techniques in comparison to discrete-event system approaches is summarised below:

- a** The burden to manage the index set that is used to take account of the behaviour of contacts on different levels such as position, velocity and acceleration for stick-slip transitions etc. is essentially reduced.
- b** The impacts, that may occur with or without collisions e.g. Painleve Paradox, velocity jumps due to C^0 constraints are a strong incentive to describe the mechanical systems as measure-differential inclusions and the MDI representation is more consistent.
- c** Systems which are zeno (systems that exhibit infinitely many switchings in finite time such as jumping ball on the ground) are problematic for event-driven schemes, where in the MDI framework they are handled realistically.
- d** By the validity of the measure-differential inclusion at even instants of discontinuity and its

ability to integrate set-valued control and set-valued force laws renders every instant to a possible transition instant.

This representation necessitates the investigation of the optimal control of measure-differential inclusions.

1.2 Optimal Control and Hybrid Systems

The frontier of variational calculus has been extended with the advent of optimal control and its versatile requirements, that pushed the development of variational techniques forward. The most important result during the last century in optimal control has been Pontryagin's Maximum Principle which has been published by Pontryagin *et al.* in [74]. In the works of Fillipov such as in [35], [36], [37] differential equations have been extended to differential inclusions. The concept of Measure Differential Inclusions is first encountered in the works of Moreau. The field of nonsmooth mechanics provided a fruitful playground where the applications of nonsmooth variational techniques flourished. The application of the measure-differential inclusion concept to the unilaterality of rigid-body mechanics is developed in the works of J.J. Moreau [64], with the added feature that he combined differential inclusions with convex analysis. The concept of measure-differential inclusion is also treated in the works of Schatzman [91] and [92]. At the same time Rockafellar, extended by its fundamental work convex analysis the horizon of variational calculus. In a series of publications such as [76], [77], [79], [82], [83] he treated many aspects of optimal control and variational analysis from the point of view of convex analysis. The concept of convexity replaced that of linearity in his works systematically. Clarke, provided in his dissertation the tools for the extension of many principles to nonconvex lipschitzian setting of optimisation which can be traced in his publications such as [23], [24], [25], [26], [27], [28] and [30]. The books of Clarke, one which he has coauthored, [29] and [31] provide excellent survey and introduction to the field of nonsmooth analysis and its application to control theory. The nonlipschitzian, extended-valued analysis has been pushed forward by Rockafellar in his publications [84], [85], [87] and [89]. In these works of Rockafellar, the subdifferential calculus is extended from the locally Lipschitzian setting to extended-real valued functions, on general linear topological spaces, such that the infinite-dimensional case is included. Optimal control of impulsive systems inevitably entails optimal control with discontinuous states. Historically, in the classical calculus of variation the well-known Weierstrass-Erdmann corner conditions are fulfilled at times where there is a discontinuity in the velocity of a system described by first-order differential equations. This approach is used in order to analyze optimality conditions on mechanical impacts by Yunt *et al.* in [113].

The main issue in the optimal control of hybrid mechanical systems has been the blending of impact mechanics with impulsive optimal control. Indeed, the optimal control of such sys-

tems entails unavoidably impulsive control. In what follows next, the new concepts required to deal with this specific problem are introduced shortly. First of all a new concept of partial integration of differential measures is required. In the framework of integration theory, this has long been recognised as a problem if state and costate should become concurrently discontinuous as has been addressed by Moreau [64] and Rockafellar [76]. Rockafellar studied in [76] the discontinuity of the dual state in constrained convex optimal control problems but dispensed of attacking the problem of concurrent discontinuity of state and costate. Moreau gave in [64] partial integration formulas for differential measures in general bilinear forms. In [68] Murray studies the extension and existence theorems of problems in calculus of variation to the setting when impulsive controls are applied and state discontinuity occurs. He bases his work on [76], and outlines in his motivation that jumps in the states may occur due to constraints on the dual dynamics which are reached by the costate, as economics as a field of application in mind. In [18] Bressan studies several classes of impulsive Lagrangian systems. The main focus is impulses generated by sudden parameter changes such as inertial parameters that affect the momentum balance, or impulses arising due to structure of constraints of a mechanical system. A certain class of impulsive systems that resemble discontinuous diffusion processes are treated in [12] by Bensoussan. In [112] by Yunt, the concept of internal boundary variations are introduced, from the point of view that the instant of state discontinuity constitute an internal boundary in the optimal control problem. The necessity that at a location of transition several conditions have to be fulfilled, gives rise to the idea of some sort of transversality conditions if one begins to consider an instant of discontinuity as a two-sided boundary where to arcs are "joined" discontinuously. So the concepts of internal boundary variations and discontinuous transversality conditions are given birth to, which are twins in some sense. They subsume the classical boundary variations and transversality conditions naturally, since the classical ones constitute the unilateral versions of bilateral new concepts. A fundamental difference in various approaches in modeling of state discontinuity is the character of the interval on which the discontinuity and the impulsive control action acts on the system. A common approach is to assume that the discontinuity happens on a Lebesgue negligible interval whereas in the other approach the interval of discontinuity is opened and a transition dynamics is implanted. In the realm of impulsive optimal control, both approaches are represented. There are a wide variety of articles devoted to the analysis of the impact phase in the optimal impulsive control, where the "contact stiffness" is taken very high, such that a rigid-body behaviour is attained in the limit. In the references by Galyaev [41], Bentsman *et al.* [60] and the book by Miller [59] the discontinuous time change approach is the underlying technique. By using the special transformation of the time scale, the method enables the conversion of an impulsive optimal control problem to standard one with ordinary differential equations, which is formulated for some auxiliary system described by ordinary differential equations with bounded controls. Re-

cently, Bentsman and Miller *et al.* and Miller and Bentsman, further complicated the analysis by considering the dynamics that may be induced through the interaction with the control strategy such as impulsive sliding modes and zeno systems, and propose it as an alternative approach to the complementarity approach in [13] and [58]. In [67] by Murphey some aspects of nonsmooth dynamics of mechanical systems and control aspects are treated. There Murphey introduces the terminology *multiple-model systems*, referring to the fact that through contact interactions such as stick-slip transitions, the mechanical system can be represented by different differential equations depending on the state of the contacts. His approach falls generally into the approach of discrete-event system analysis in hybrid system terminology. The transitions between the models are on a set of measure zero. The author focuses on the stability analysis of systems with friction transitions. In the approaches provided in references such as [8], [48] and [96], the impulsive control problem is transformed into a problem of an ordinary differential inclusion problem, which requires to determine trajectories for the "discontinuous" states during the "impulsive" control action. Though the modeling approach depends on the task, this approach has two main disadvantages. Hybrid system theory is based mainly on the assumption that the transitions happen instantaneously, and the main modeling approaches are based on the instantaneous transition concept. The second disadvantage lies in the inability of this interval opening approach to tackle with the combinatorial nature of the mode sequencing and transition time and location determination problem. Philosophically, in order to resolve this combinatorial problem properly every instant must be equipped with ability to have the potential to switch to all other possible modes, which is practically not possible in the interval opening approach. In [60], impulses arising from unilateral constraints are considered but again in the framework of interval-opening approach and transformation technique. The instantaneous transition approach is in comparison to other impulsive necessary conditions consistent with different common hybrid system modeling methods in which transitions happen instantaneously such as in [17]. In [40] Galbraith and Vinter discuss the optimal control of hybrid systems with an infinite set of discrete states in an abstract setting. An event driven approach to hybrid dynamical systems is used similar to [17]. The properties of the resulting value function is characterised. The authors claim that the analysis is extendable to optimal control of dynamical systems with discontinuous states by invoking bounds on the transition instants. Egerstedt *et al.* discuss in [33] and [34] a numerical algorithm based on the calculation of the gradient of the value function with respect to switching times and propose a gradient descent based algorithm that determines suboptimal solutions and the method is for continuous state trajectories which are Lipschitz. In [102] Verriest *et al.* consider an impulsive optimal control problem with discontinuity in the system state and delay response to impulsive input. Necessary conditions are stated in first-order form by making use of the Hamiltonian concept. The instant of impulsive control action and transition locations are free, however, the variations at

the internal boundaries of the time domain where the system state is discontinuous seemingly only derived by the assumption that the variations of the pre-, and post-transition states are having only the component for fixed impulse time, which seems contradictive to the proposed problem. In two publications [106] and [107] Xu and Antsaklis present a direct numerical method for the switching time optimisation for systems without state discontinuity at transitions. Shaikh presents in [93] and [94] necessary and sufficient conditions for hybrid dynamical systems with state-continuous transitions by making use of needle variation technique. In [104] considers the optimal control of impulsive dynamical systems, which are exposed at fixed time-instants to impulses that are generated autonomously. The magnitude of the impulsive force is dependent on the system state but the controls are ordinary and nonimpulsive. In a series of publications Ahmed [2], [3], [4] and [5] studies existence and necessary conditions for the optimal control of measure-differential equations of the form:

$$d\mathbf{x} = \mathbf{A} \mathbf{x} dt + f(\mathbf{x}) dt + \mathbf{g}(\mathbf{x}) \boldsymbol{\nu}(dt), \quad \mathbf{x}(0) = \boldsymbol{\xi}, \quad t \geq 0. \quad (1.1)$$

He discusses the existence and regularity of the solutions and relates them to optimal control. There are some works devoted to the complementarity modeling and optimal control of mechanical systems as given in references [109], [114], [115], [116]. In [20] Brogliato studies the problem of quadratic optimal control of unilaterally constrained linear time-invariant systems. His motivation is to define a class of optimal control problems for which the higher-order Moreau's sweeping process constitutes the numerical resolution of its necessary conditions. Optimal control of mechanical systems with unilateral constraints has been analysed case dependent in several publications, such as in [108], where an optimal control strategy for the high jump of an hopper is studied based on complementarity modeling.

In this work, it is assumed that the instant of discontinuity is reduced to an instant with Lebesgue measure zero, instead of taking an interval opening approach, which is the approach considered in literature so far.

1.3 Internal Boundary Variations and Discontinuous Transversality Conditions

The approach taken in this thesis, before being converted into variational necessary conditions, is first expounded in its philosophical approach. A transition with a discontinuity in the state can be regarded as an internal boundary in the domain of interest. Historically, the Weierstrass-Erdmann corner conditions were among the first variational conditions to deal with trajectories with a corner. The idea of internal boundary variations and discontinuous transversality conditions emanated from the motivation to provide variational criteria for time and location, at the instants where impulsive transitions happen under constraints and discontinuity in the state.

An early attempt of the author is given in [113] where under relatively restrictive assumptions and using Weierstrass-Erdmann corner conditions optimality conditions for mechanical impacts are investigated.

A time instant of Lebesgue measure zero is considered as a transition time $t_i \in \mathcal{I}_T$ if one of the two events occur together or for itself:

- **Event 1** Some directions of motion of the system are opened or closed by the control strategy, which entails a change in the degrees of freedom (DOF) of the system.
- **Event 2** An impulsive control action is exerted on the system, which may be accompanied by a discontinuity of the generalised velocities of the Lagrangian system.

Here $\mathcal{I}_T$ denotes the set of transition instants of the process.

The concurrence of both events where some directions of motion are closed is called "blocking". In the time-optimal control of dynamical systems one has to consider the variations in the end time. In the classical calculus of variations where the final state and final time are free, the variations of the final state are composed of two parts, namely, the part that arises of the variations at a given time and the part arising from variations due to final time. Since the transitions times are assumed to be free, the two-part character of the variations at pre-, and post-transition states is considered. The assumptions during a possibly impactive transition are given as follows:

- The transitions may be impactively.
- The generalised position remain unchanged during transition.
- The impulsive control action acts on the system at a time instants t_i which are Lebesgue-negligible and are countably many.
- At a possibly impactive transition, the pre-transition controller configuration is assumed to be effective.
- There are no transitions at initial time t_0 and final time t_f .

The assumption that impulsive forces are Dirac distributions, enable their consideration on an atomic instant of time, since the integral of the Dirac distribution is constant, irrespective of the measure of its support. The above stated assumptions are converted into requirements to the variations at the internal boundaries. At the boundaries of the time domain, the pre-transition state variations are considered separately from the post-transition variations. In impact mechanics, the generalised accelerations and velocities are eligible to become discontinuous where as the generalised positions are of absolutely continuous character. The absolute continuity of the generalised positions means that the total variation of the generalised positions at the

pre-transition and post-transition instants are equal. The pre-transition and post-transition variations are interrelated by the transition conditions which can be seen as the fundamentals of transversality conditions that join two trajectories discontinuously. The transition conditions are introduced symmetrically with respect to pre-, and post-transition states. The transition conditions are of two types, namely, the impact equations and the constitutive impact laws. The impact equations relate the discontinuity in the impulse of the Lagrangian system to the impulsive forces/controls. The impact law (.i.e. the moreau-newton impact law), however, is a constitutive law which is chosen depending on the modeling approach preferred. As a case study, in reference [115] by Yunt the blocking of some DOF of an underactuated manipulator by tangential fully-inelastic impact is discussed and the necessary conditions are stated.

1.4 Numerical Analysis

The literature on the numerical techniques that have found application in the trajectory optimisation and optimal control of dynamical systems is vast. The numerical methods in optimal control are divided in two broad classes, namely, direct methods and indirect methods. The direct methods comprise numerical approaches where after a suitable discretisation method for the dynamics the goal function is minimised directly. These methods use only control and state variables as optimisation variables and dispense completely with adjoint variables. The discretised adjoint states are obtained by a post-optimal calculation using the dual multipliers of the resulting nonlinear programming problem. In indirect methods one has also the necessary conditions which have to be fulfilled. The references [15] by Betts and [22] Büskens *et al.* provide a general overview on the methods that have been used so far if dynamical processes are modeled as differential equations with constraints among many others. The theory of necessary conditions for optimal control problems with control and state constraints has been developed in the second part of the last decade. The theory at hand treats optimal solutions as solutions of multi-point boundary value problems (MBVP). For this class of MBVP's, shooting techniques have been developed as efficient and reliable numerical methods providing highly accurate solutions. However, these methods need a close enough initial guess of the optimal state, control and adjoint variables and require a detailed a priori knowledge of the structure of the optimal solution, such as the number of active time intervals for each of the constraints. In practice, it is difficult to determine the structure of the optimal control and to find appropriate estimates for the adjoint variables a priori.

When considered from the perspective of discretisation, there are two major approaches to handle the optimal control of finite-dimensional Lagrangian systems in the trajectory optimisation numerically. These are based on time-stepping (sweeping) and event-driven approaches. Event-driven approaches lead to optimisation problems that belong to the class of mixed integer

programming problems and branch and bound techniques are methods used in order to search for the minimum. In this thesis, however, the time-stepping approach is considered, which is the natural numerical extension of the MDI representation of the dynamics.

Numerically, the treatment of the optimal control of FDLS is related to the applied mathematics branch called Mathematical Programming of Equilibrium Constraints. In literature there are numerous methods for trajectory optimisation of hybrid dynamical systems. In the framework of this thesis, they are investigated in two broad classes, namely, switching finite-dimensional Lagrangian systems and hybrid finite-dimensional Lagrangian systems. In [73] by Outrata *et al.* a MPEC is defined as an optimisation problem in which the essential constraints are defined by parametric variational inequality or complementarity systems. One of the many representations of a MPEC can be stated as follows:

$$\min_{\mathbf{x}, \mathbf{z}} f(\mathbf{x}, \mathbf{z}), \quad (1.2)$$

$$\mathbf{z} \in \mathcal{S}(\mathbf{x}), \quad (1.3)$$

$$\mathbf{x} \in \mathcal{U}_{ad}, \mathbf{z} \in \mathcal{Z}. \quad (1.4)$$

The problem described by (1.2), (1.3) and (1.4) includes a subclass of so-called bilevel programs, where $\mathcal{S}$ assigns each $\mathbf{x} \in \mathcal{U}_{ad}$ the necessary conditions of a "lower-level" optimisation problem. In the case where the complementarity system arises from mechanical systems without Coulomb friction, a so-called subclass of MPEC, namely, bilevel programs apply. In references [32] by Cottle *et al.* and [70] by Murty, detailed treatment of complementarities and optimisation can be found. References [57] by Luo *et al.* and [73] by Outrata treat MPEC and bilevel programs extensively. In the framework of this thesis, the control action is represented by $\mathbf{x} \in \mathcal{U}_{ad}$. The differential measures of control can be considered as the variables of the "higher-level" optimisation problem whereas the contact forces and states are the variables of the "lower-level" problem. By analogy, the measure-differential inclusion, that describes the dynamics as a balance of measures, can be considered as the necessary conditions of a "lower-level" optimisation problem represented by the saddle-region restraining set $\mathcal{S}$. The comprehension of the structure of structure-variant mechanical systems paves the way to suitable numerical algorithms, by considering of this unique nature of mechanical systems reveals through the extended principle of Gauss the necessary optimality conditions in complementarity and proximal form for FDLS. In [63] it has been shown that the determination of the accelerations of a mechanical system subject to unilateral constraints without friction can be cast into a primal and dual quadratic programming problem. Further, it is shown that a generalisation of the Gauss' variational principle is valid in the case of unilateral constraints without friction. In [44], it is shown that a quadratic programming problem can be obtained if Tresca type friction, for which the normal force is decoupled from the tangential force, exists and that the equations of motion along with the linear-complementarity conditions constitute necessary Karush-Kuhn-Tucker conditions of

optimality for the quadratic programming problem. If Coulomb type friction exists at the contacts, then the optimal control problem is subject to variational inequalities (VI) and there does not exist a QP of which solution is equivalently representable by the resulting VI.

There are several historical corner stones in the numerical analysis of complementarity systems. The study of complementarity problems is a flourishing field since its advent at the beginning of the sixties. In the beginning, the linear complementarity problem drew the attention because of the structure of the Karush-Kuhn Tucker conditions of a general Quadratic Programming (QP) Problem, which has a LCP structure. The formulation of the computation of the Nash equilibrium point of a bimatrix game as a LCP by Howson and Lemke developed an efficient pivoting algorithm, the complementary pivot method, marked the establishment of the LCP's as a branch of applied mathematics of its own. In 1968, came the unification of linear and quadratic programming problems and bimatrix games in the LCP framework by Cottle and Dantzig. The nonlinear complementarity problem (NCP) has been introduced by Cottle in his doctor of philosophy thesis in 1964. The concept of Variational Inequality Problem has been defined in 1966 by Hartman and Stampacchia in the framework to compute stationary points of nonlinear programs.

The numerical optimisation in this thesis for hybrid Lagrangian trajectories rely on the augmented Lagrangian method which has been extensively investigated by R. T. Rockafellar in his works such in [80], [81] and [86]. There are several aspects that make the augmented Lagrangian approach favorable in comparison to other approaches. In exact penalty methods, instead of performing a sequence of minimisations, a single minimisation is performed but the penalty parameter has to be set very high, such that ill-conditioning is caused, and depending on the minimisation method used the condition number of the Hessian matrix is severed. The sequential minimisation besides preventing ill-conditioning, allows partial minimisations especially at the initial stages of the successive minimisations such that the successive minimisations proceed faster than expected. The global optimizing property as in the exact penalty approach is preserved. The Newton method in the minimisation of unconstrained functions is a favored method because of its superlinear convergence in the vicinity of the solution. However, in large scale optimisation problems where the structure of the Hessian matrix is not sparse, the evaluation of the Hessian matrix becomes cumbersome. In the case of the augmented Lagrangian technique, in general, by the convexification induced by adding quadratic penalty terms to the goal function, the structure of the Hessian matrix becomes dense. In such cases, quasi-Newton methods are used in order to extrapolate the Hessian matrix by a formula like BFGS method. If the function to be minimised is nonsmooth, then the nonuniqueness and unboundedness of the Hessian may cause problems. A way to circumvent the problem of the nonuniqueness and unboundedness of the Hessian matrix often induced by using reformulation functions is to use smoothing. This has the disadvantage however that the smoothing parameter has to be

adjusted in the course of the optimisation, which reduces the speed of the algorithm as well. The modified conjugate gradient methods offer a trade-off by performing a minimisation based on second-order information without calculating them. The only required information is first-order information such as an element of the subgradient and second-order information such as estimates of the Hessian are extrapolated numerically, and it is not necessary to calculate the dense Hessian matrix.

The concept of controlling hybrid systems requires to make decision of discrete decisions. The dependence of the optimisation problem to decisions is treated in the framework of sensitivity analysis. As can be expected, the sensitivity of a numerical treatment of an optimal control problem to discrete decisions is high. In order to accomplish the sensitivity analysis the value function is discussed in the nonsmooth setting and is related to the augmented Lagrangian approach. Lagrange multipliers are the measure for the sensitivity of an optimisation problem to a given set of constraints. Classically, Lagrange multipliers were viewed as auxiliary variables, which were introduced in a constrained minimisation problem in order to write first-order optimality conditions as a system of equations. The needs arising from more complicated constraint structures required an in-depth understanding of dual variables, and they have been characterised by the methods of nonsmooth geometry, which makes use of one-sided tangent and normal vectors to a set of points that satisfy the constraints. The modern approach to Lagrange multipliers is presented in [27] and [90].

The classical theory of optimisation presumes certain differentiability and strong regularity assumptions. However, these assumptions are too demanding for many practical applications, since functions involved are often nonsmooth, that is, they are not necessarily differentiable. The source of the nonsmoothness may be the objective function itself, its possible interior function, or both. Moreover, there exists so-called stiff problems that are analytically smooth but numerically nonsmooth. This means that the gradient varies too rapidly, and, thus, these problems behave like nonsmooth problems. The methods for nonsmooth optimisation can be divided into two main classes, namely, subgradient methods and bundle methods. Both approaches are based on the assumption that functionals to be minimised are locally Lipschitz continuous. This assumption is necessary because of the requirement to evaluate the functional and one of its arbitrary subgradients in every point of the domain of interest. The problem, thus, need not to be differentiable or convex. The subgradient methods have mainly developed in the 60s in the Soviet Union and an excellent overview is given in [95]. Their basic idea is to generalize the methods for smooth problems by replacing the gradient by an arbitrary subgradient. The main handicap for these methods is the absence of an implementable stopping criterion. The fact that the direction opposite to an arbitrary subgradient need not yield a direction of descent, excludes them from the class of descent methods. The second class of methods are the bundle methods. The main idea is to exploit the previous iterations by gathering the subgradient

information into a bundle. The first bundle method, the so-called ϵ -steepest descent method was developed in Lemaréchal in [54]. It was based on the conjugate subgradient method by Lemaréchal [53] and Wolfe [105]. In [49] Kiwiel gave a new approach to bundle methods, which was based on the classical cutting plane method. The main idea in this method is to form a convex piecewise linear approximation to the objective function using the linearisations generated by the subgradients.

The thesis is structured as follows:

Chapter 1 provides an introduction to this work. In this chapter the reasons, why this problem is attacked, is presented. Further, it shall give the reader an overview on the character of problem being handled. It also provides historical background of vast areas of research such as calculus of variations and optimal control, measure theory, set-valued analysis, nonsmooth analysis, variational inequalities and complementarities, hybrid system computation and control on a rather limited space with emphasis on key issues, persons and events. This shall enable the placement of this work in a proper place in a long research history. In the introduction in a comparative manner, the differences to other works done in this field is provided. These comparisons include approaches in numerical issues as well as theoretical considerations.

Chapter 2 deals with the fundamental issue of modeling impulsively controlled FDLS. The modeling is a fundamental attribute that shapes the optimisation approach extensively. The chapter begins with the analysis of projected Newton-Euler Equations in impulsive control form which is first presented in [110]. In this context the concept of set-valued, impulsive and unbounded controls is introduced. A measure-differential inclusion (MDI) based modeling approach for finite-dimensional Lagrangian systems is introduced, that can exhibit autonomous or controlled mode transitions, accompanied by discontinuities on velocity and acceleration level. The introduced framework has the ability to model and control hybrid mechanical systems with discontinuous transitions among different system modes. Modeling of FDLS as Linear Complementarity Problem (LCP) is presented. The definition of hybrid finite-dimensional Lagrangian system is given in association with this modeling approach. A general measure-differential inclusion representation of finite-dimensional Lagrangian systems is derived, in which the set-valued inclusions are represented as linear complementarities. The aim is to extend the existing modeling framework such that controls of impulsive and set-valued character are fit in. Starting from the Lagrangian formalism in impulsive control form, the modeling is further detailed in the successive section to include a linear complementarity structure, that enables to consider the interaction of impulsive controls with together with set-valued impulsive interactions of the system with the surroundings. The Lagrangian formalism in impulsive control form is already discussed in [110]. The main part of the complementarity modeling of controlled complementarity mechanical system is published in [117]. The model in chapter 2 about modeling complementarity systems is enriched by fitting in properly the measure theory, and extending

the modeling on acceleration level, on impactive form and measure-differential inclusion form to some suitable quadratic problems which can be seen as variants of the extended problem of Gauss and its dual, which first has been discussed in framework of QP by [63] for rigid body mechanical systems of which dynamics are represented by differential equations subject to constraints on position level. The existence of such quadratic programs in different cases is in the opinion of the author important for numerical considerations. The idea to consider such variants of the extended problem of Gauss and its dual is first mentioned in [115]. Further in [115], the complementarity representation of blocking is used, for which a detailed discussion and derivation is provided in chapter two. The chapter concludes with a case study, in which the modeling framework for underactuated scleronomic mechanical systems with impactively blockable DOF is presented. This modeling framework is already discussed in [111], and is supplemented here further by the analysis of dissipated energy, change of kinetic energy and of the impulse of the system.

Chapter 3 is devoted to the numerical methods in calculating optimal trajectories. There are two numerical methods that are being introduced in this chapter, one nonlinear programming approach and a shooting method. Both methods reside on augmented Lagrangian techniques, because of its efficiency in tackling with nonconvex optimisation problems. After introducing the preliminaries of nonsmooth analysis and differentiation in finite-dimensional spaces, the local and global properties of the augmented Lagrangian method is discussed. The properties of the augmented Lagrangian method are analysed by making use of the techniques that derive from nonsmooth geometrical considerations. The introduced methods are direct methods. Two mechanical models are investigated in detail. In the NLP case the global optimal trajectories of an underactuated double pendulum with impulsively blockable DOF is presented. In the analysis of the shooting method, a five-DOF differential drive model is used to present the properties of the shooting method.

Chapter 4 deals with the derivation of necessary conditions for the impulsive optimal control problem in second-order form, since dynamic mechanical systems have a representation in second-order differential form. The main result of this chapter is published in [112]. In this chapter the discontinuous transversality conditions and internal boundary variations are expounded in detail. Necessary conditions for the impulsive time-optimal control of finite-dimensional Lagrangian systems is stated. The derivation is based on the application of subdifferential calculus techniques, to lower-semicontinuous extended-valued functionals. The necessary conditions for the impulsive time-optimal control of underactuated manipulators with impulsively blockable DOF is provided. The concepts of internal boundary variations and discontinuous transversality conditions are introduced in detail in this chapter.

Chapter 5 is devoted to the derivation of the same necessary conditions as in Chapter 4 in the first-order form, which gives access to the well-known Hamiltonian framework. The

subdifferential calculus for extended-real valued functions, on general linear topological spaces as developed by Rockafellar is the main instrumentation in the derivation of the necessary conditions. Further, a hybrid system definition of finite-dimensional Lagrangian systems is given, and a hybrid impulsive minimum principle is stated. The necessary conditions for the time-optimal control of underactuated manipulators with blockable DOF is stated in a first-order system setting. The main of the results are already published and presented in [110]. Necessary conditions are stated in the first-order form, and by considering upper directional derivatives of the Hamiltonian in different phases of motion, the structure of the dual state is characterised.

Chapter 6 concludes with a discussion of the results and future directions of work. The contributions of the PhD Thesis can be summarised in several points:

Claims of Originality

- Establishment of the relationship between finite-dimensional Lagrangian systems (FDLS) and Hybrid System Modeling and Control within the Measure Differential Inclusion framework
- The definition and introduction of internal boundary variations and discontinuous transversality conditions as generalisations to their classical counterparts
- The relation of the Discontinuous Transversality Conditions to impulsive optimal control
- Proposition of a NLP algorithm and shooting method for the analysis of Hybrid Lagrangian systems based on Moreau's sweeping process
- Complementarity characterisation of blocking as a set-valued control law
- Statement of Necessary Impulsive Optimal Control Conditions for structure-variant finite-dimensional Lagrangian systems in Second-Order Form
- Statement of Necessary Impulsive Optimal Control Conditions for structure-variant finite-dimensional Lagrangian systems in First-Order Form
- Application of the derived concepts to the control of underactuated manipulators with impulsively blockable DOF
- The general development of a modeling framework for the impulsive control of hybrid finite-dimensional Lagrangian systems
- The introduction and development of a single-shooting method for switched Lagrangian systems based on the augmented Lagrangian method and time-stepping integration.

- The introduction and development of a direct method that is based on an augmented Lagrangian based nonlinear programming scheme for the determination optimal trajectories of finite-dimensional impulsive hybrid Lagrangian systems.
- The development of an augmented Lagrangian based MPEC optimisation scheme that performs the minimisation based on modified conjugate gradients methods.
- The combinatorial NP-hard nature of mixed-integer programming approaches is circumvented and a polynomial time methods is developed that is able to choose a sequence of transitions and modes.

This work is based on following publications:

Published and Submitted Publications

- K. Yunt and Ch. Glocker, Trajectory Optimisation of Hybrid Mechanical Systems using SUMT, *IEEE Proc. of Advanced Motion Control*, Istanbul, 2006, pp 665-671.
- K. Yunt and Ch. Glocker, Time-Optimal Trajectories of a Differential-Drive Robot, *Proceedings of the Fifth Euromech Nonlinear Dynamics Conference*, D. H. Van Campen, M. D. Lazurko, W.P.J.M van der Oever (Eds), pp. 1589-1596.
- K. Yunt and Ch. Glocker, A Combined Continuation and Penalty Method for the Determination of Optimal Hybrid Mechanical Trajectories, *IUTAM Symposium on Dynamics and Control of Nonlinear Systems with Uncertainty 2006*, H.Y. Yu and E.Kreuzer (Eds.), IUTAM Bookseries, Springer, 2007, pp. 187-196.
- K. Yunt, Optimal Trajectory Planning for Structure-Variant Mechanical Systems, *IEEE Proc. of Int. Workshop on Variable Structure Systems VSS'06*, 2006, pp 298-303.
- K. Yunt and Ch. Glocker, Modeling and Optimal Control of Hybrid Rigid-Body Mechanical Systems, *Hybrid System Computation and Control (HSCC07)*, A. Bemporad, A. Bichi and G. Buttazzo (eds.), Springer, Lecture Notes in Computer Science (LNCS) 4416, 2007, pp 614-627.
- K. Yunt, Impulsive Time-Optimal Control of Underactuated Manipulators with Impactively Blockable Degrees of Freedom, *European Control Conference (ECC'07)*, 2007, pp 3977-3984.
- K. Yunt, Impulsive Time-Optimal Control of Structure-Variant Rigid-Body Mechanical Systems, 3rd International IEEE Scientific Conference on Physics and Control (Physcon 2007), 2007 IPACS Open Library [:/lib.physcon.ru](http://lib.physcon.ru).

- K. Yunt, Necessary Conditions for Impulsive time-optimal Control of Finite-Dimensional Lagrangian Systems, *Hybrid System Computation and Control (HSCC08)*, M. Egerstedt and B. Mishra (eds.), Springer, Lecture Notes in Computer Science (LNCS), pp. 556-569.
- K. Yunt and Ch. Glocker, Optimality Conditions for Mechanical Impacts, *Proceedings of Applied Mathematics and Mechanics*, GAMM 2004, Vol.4, pp. 630-631.
- K. Yunt, Necessary Conditions for the Impulsive Optimal Control of Multibody Mechanical Systems, *Zeitschrift für Angewandte Mathematik und Mechanik*, (submitted on March, 15th 2008)
- K. Yunt, Necessary Conditions for the Impulsive Time-Optimal Control of Rigid-Body Lagrangian Systems, *Systems and Control Letters*, (submitted on September, 24th 2007)

Chapter 2

Modeling of Hybrid Finite-Dimensional Lagrangian Systems

2.1 Preliminaries

All the vectors are considered as column vectors and, correspondingly, all the transposed vectors are considered as row vectors. The usual inner product is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\|\mathbf{x}\|$ represents the Euclidean norm.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i \quad (2.1)$$

$$\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}} \quad (2.2)$$

where $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$ and $x_i, y_i \in \mathbb{R}$ are the i th components of the relating vectors.

Definition 2.1.1 -Indicator Function The indicator function $\Psi_{\mathcal{C}}(\mathbf{x})$ of a convex set $\mathcal{C}$ takes the value zero if $\mathbf{x} \in \mathcal{C}$ and infinity otherwise,

$$\Psi_{\mathcal{C}}(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{C} \\ \infty & \text{if } \mathbf{x} \notin \mathcal{C}. \end{cases} \quad (2.3)$$

Definition 2.1.2 -Conjugate Function The conjugate function $f^*(\mathbf{x}^*)$ of a convex function $f(\mathbf{x})$ is defined as:

$$f^*(\mathbf{x}^*) = \sup_{\mathbf{x}} (\langle \mathbf{x}, \mathbf{x}^* \rangle - f(\mathbf{x})). \quad (2.4)$$

Definition 2.1.3 -Subdifferential of a Convex Function The subdifferential $\partial f(\mathbf{x})$ of a convex function $f(\mathbf{x})$ is a set containing the gradients of all supporting hyperplanes of $f(\mathbf{x})$,

$$\partial f(\mathbf{x}) = \{\mathbf{y} \mid f(\mathbf{x}^*) \geq f(\mathbf{x}) + \langle \mathbf{y}, \hat{\mathbf{x}} - \mathbf{x} \rangle; \quad \forall \mathbf{x}^* \}. \quad (2.5)$$

Definition 2.1.4 -Normal Cone The normal cone $\mathcal{N}_{\mathcal{C}}(\mathbf{x})$ to a convex set $\mathcal{C}$ is the set of all vectors that have an obtuse angle with all vectors $\mathbf{x}$ emanating from $\mathcal{C}$ to any $\hat{\mathbf{x}} \in \mathcal{C}$,

$$\mathcal{N}_{\mathcal{C}} = \{\mathbf{y} \mid \langle \mathbf{y}, \hat{\mathbf{x}} - \mathbf{x} \rangle \leq 0; \quad \forall \hat{\mathbf{x}} \in \mathcal{C}; \quad \forall \mathbf{x} \in \mathcal{C}\}. \quad (2.6)$$

The normal cone to a convex set $\mathcal{C}$ at the point $\mathbf{x} \in \mathcal{C}$ is given by $\mathcal{N}_{\mathcal{C}}(\mathbf{x})$ in the sense of convex analysis [82].

Definition 2.1.5 -The Linear Complementarity Problem [70] Given $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{b} \in \mathbb{R}^n$ find $\mathbf{w} = (w_j) \in \mathbb{R}^n$, $\mathbf{z} = (z_j) \in \mathbb{R}^n$ satisfying

$$\mathbf{w} - \mathbf{A} \mathbf{z} = \mathbf{b}; \quad \mathbf{w}, \mathbf{z} \succeq 0; \quad \langle \mathbf{w}, \mathbf{z} \rangle = 0. \quad (2.7)$$

This LCP is denoted by the tuple $(\mathbf{b}, \mathbf{A})$.

Lemma 2.1.1 The subdifferential of the indicator function is the inverse of the subdifferential of the support function,

$$\mathbf{x}^* \in \partial \Psi_{\mathcal{C}}(\mathbf{x}) \quad \Leftrightarrow \quad \mathbf{x} \in \partial \Psi_{\mathcal{C}}^*(\mathbf{x}^*). \quad (2.8)$$

Lemma 2.1.2 The basic complementarity relation

$$x \geq 0, \quad y \geq 0, \quad xy = 0, \quad (2.9)$$

can equivalently be represented as:

$$x \in \mathcal{N}_{\mathbb{R}_0^+}(y) \quad (2.10)$$

or

$$y \in \mathcal{N}_{\mathbb{R}_0^+}^*(x). \quad (2.11)$$

Lemma 2.1.3 The set-valued Signum relation $y \in \text{Sgn}(x)$

$$y = \begin{cases} 1 & \text{if } x \in \mathbb{R}^+, \\ [-1, 1] & \text{if } x = 0, \\ -1 & \text{if } x \in \mathbb{R}^-. \end{cases} \quad (2.12)$$

can be decomposed into two UPR relations:

$$y_L \in \text{Up}(x_L) - 1 \quad (2.13)$$

$$y_R \in -\text{Up}(x_R) + 1 \quad (2.14)$$

where

$$y = y_L + y_R, \quad x = x_R - x_L. \quad (2.15)$$

The generalisation of the fundamental LCP is the mixed LCP (mLCP).

Definition 2.1.6 -The Mixed Linear Complementarity Problem [70] Given $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{m \times m}$, $\mathbf{C} \in \mathbb{R}^{n \times m}$, $\mathbf{D} \in \mathbb{R}^{m \times n}$, $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, find $\mathbf{v} \in \mathbb{R}^m$, $\mathbf{u} \in \mathbb{R}^n$ satisfying

$$\mathbf{a} + \mathbf{A} \mathbf{u} + \mathbf{C} \mathbf{v} = \mathbf{0}, \quad \mathbf{b} + \mathbf{D} \mathbf{u} + \mathbf{B} \mathbf{v} \geq \mathbf{0}, \quad \mathbf{v} \geq \mathbf{0}, \quad \langle \mathbf{v}, \mathbf{b} + \mathbf{D} \mathbf{u} + \mathbf{B} \mathbf{v} \rangle = 0. \quad (2.16)$$

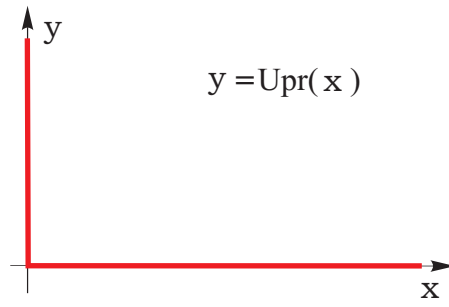


Figure 2.1: Unilateral primitive.

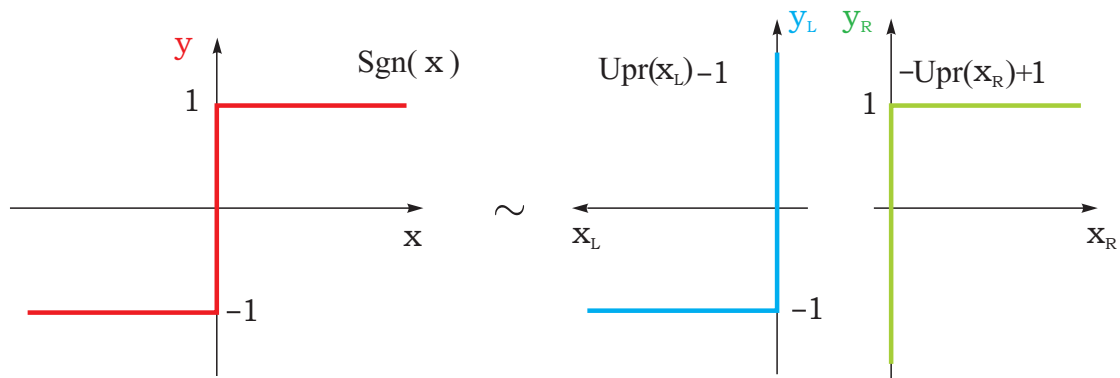


Figure 2.2: Decomposition of the Signum relation into two Uprs.

The right-continuous and left-continuous regularisations of a function f , which is a mapping of $\mathcal{I}$ to a Hausdorff topological space $\mathcal{E}$; becomes important if one considers that for every $t_i \in \mathcal{I}_T$ the right-side limit given by:

$$f^+(t_i) = \lim_{s \rightarrow t_i, s > t_i} f(s) \quad (2.17)$$

may differ from $f(t_i)$, if it exists. Symmetrically, the left-side limit, if it exists, is denoted by $f^-(t_i)$. Following proposition is used often in this work:

Proposition 2.1.1 [64] Let $\mathcal{E}$ be regular and let $f : \mathcal{I} \rightarrow \mathcal{E}$ be such that for every $t \in \mathcal{I}$ different from the possible right end of $\mathcal{I}$, there exists $f^+(t)$; then

$$f^+(t) = \lim_{s \rightarrow t, s > t} f^+(s). \quad (2.18)$$

If, in addition, for every t different from the possible left end of $\mathcal{I}$, there exists ; then

$$f^-(t) = \lim_{s \rightarrow t, s < t} f^-(s). \quad (2.19)$$

As short hand notation one has:

$$(f^+)^+ = f^+, \quad (f^-)^+ = f^+, \quad (f^-)^- = f^-, \quad (f^+)^- = f^-. \quad (2.20)$$

Properties of Bounded Variaton Functions Every function of bounded variation $\mathbf{u}$, in this case the generalised velocities of the mechanical system, is associated with an $\mathbb{R}^n$ -valued regular Borel measure $d\mathbf{u}$ on $[t_0, t_1]$. The atoms for $d\mathbf{u}$ occur only at discontinuities of $\mathbf{u}$, of which there are at most countably many. Trajectories of bounded variation in $\mathbb{R}^n$ are defined to be an equivalence class, and the space of all arcs is denoted by $\mathcal{BV}$. The space of absolutely continuous arcs $\mathcal{AC}$ is a subspace of $\mathcal{BV}$. There are uniquely determined functions $\mathbf{u}^+(t)$ and $\mathbf{u}^-(t)$ in $[t_0, t_1] \rightarrow \mathbb{R}^n$, right and left continuous respectively, such that $\mathbf{u}^+(t) = \mathbf{u}^-(t) = \mathbf{u}(t)$ at all the nonatomic points, and at the end points $\mathbf{u}^-(t_0) = \mathbf{u}(t_0)$ and $\mathbf{u}^+(t_f) = \mathbf{u}(t_f)$ are valid. The quantity $\Delta\mathbf{u}(t) = \mathbf{u}^+(t) - \mathbf{u}^-(t)$ is called the jump of the arc $\mathbf{u}$ at t , and if it is nonzero there is an atom of $d\mathbf{u}$ at t with this value. Since the jumps of generalised velocities are induced by impulsive controls, these impulsive controls also occur at Lebesgue-negligible atoms and are countably many. The absolutely continuous part of the measure $d\mathbf{u}$ is denoted by $\dot{\mathbf{u}} dt$. The singular part of $d\mathbf{u}$, can be represented as $(\frac{d\mathbf{u}}{d\sigma}) d\sigma$, where $d\sigma$ is some nonnegative singular measure (a regular Borel measure), and $\frac{d\mathbf{u}}{d\sigma}$ is the Radon-Nikodym derivative of $d\mathbf{u}$ with respect to $d\sigma$, which is also denoted as $\mathbf{u}'$. For bounded variation analysis, the reference [64] is a standard reference.

2.2 The Euler-Lagrange Equations in Impulsive Control Form

The interaction of the Lagrangian system with the surroundings as well as the control actions imposed on the system necessitates to allow discontinuity events in the velocities and accelerations of the system. Let $\mathbf{q}$, $\mathbf{u}$, $\dot{\mathbf{u}}$ represent the position, velocity and acceleration in the generalised coordinates of a scleronomic FDSL with n degrees of freedom (DOF), respectively. The equations of motion (EOM) can be obtained by using the well-known Lagrange equation for the smooth dynamics :

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \mathbf{u}} \right)^T - \left(\frac{\partial T}{\partial \mathbf{q}} \right)^T + \left(\frac{\partial V}{\partial \mathbf{q}} \right)^T - \mathbf{f} = \mathbf{0} \quad \text{a.e.} \quad (2.21)$$

Here $T(\mathbf{q}, \mathbf{u})$ denotes the total kinetic, and $V(\mathbf{q})$ the total smooth potential energy of the system. The Euler-Lagrange equations have to be supplemented with some force laws that relate the external forces $\mathbf{f}$ and controls $\boldsymbol{\tau}$ with the system's state $(\mathbf{q}, \mathbf{u})$. The existence of the accelerations $\dot{\mathbf{u}}$ on an time interval are limited to the instants where the generalised velocities $\mathbf{u} \in \mathcal{LBV}$ and Lebesgue-measurable controls $\boldsymbol{\tau}$ are continuous. The time instants $\{t_j\}$ at which the discontinuities of $\dot{\mathbf{u}}$ occur are the members of the set $\mathcal{I}_D$. Because of the set of discontinuity points $\{t_i\} \in \mathcal{I}_T$ of $\mathbf{u}$ where $\dot{\mathbf{u}}$ does not exist, the Euler-Lagrange equations should be stated in the following form:

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} - \mathbf{h}(\mathbf{q}, \mathbf{u}) = \mathbf{f} + \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}, \quad \text{a.e.} \quad (2.22)$$

Here $\mathbf{M}$ is the symmetric and positive definite generalised mass matrix depending smoothly on $\mathbf{q}$, and $\mathbf{h}(\mathbf{q}, \mathbf{u})$ is a smooth function of $\mathbf{q}$ and $\mathbf{u}$ containing the gyroscopical, centripetal, coriolis accelerations as well as all other finite smooth forces such as spring and damper forces of the Lagrangian system. The linear operator $\mathbf{B}(\mathbf{q})$ includes the generalised directions of control force action on the system. The linear operator $\mathbf{M}(\mathbf{q})$ and the vector $\mathbf{h}$ are related to the Lagrangian formalism by the following equations:

$$\mathbf{M}(\mathbf{q}) = \left(\partial_{\dot{\mathbf{q}}}^2 T(\mathbf{q}, \dot{\mathbf{q}}) \right)^T, \quad \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = \left(\partial_{\dot{\mathbf{q}}}^2 T(\mathbf{q}, \dot{\mathbf{q}}) \right)^T \dot{\mathbf{q}} - \left(\partial_{\mathbf{q}} (T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})) \right)^T.$$

In order to investigate the discontinuity points of the velocities $\mathbf{u}$ and accelerations $\dot{\mathbf{u}}$ properly, equation (2.22) is replaced by the corresponding equality of measures as proposed by Moreau in [65], where it has been introduced for uncontrolled rigid-body Lagrangian systems:

$$\mathbf{M}(\mathbf{q}) d\mathbf{u} - \mathbf{h}(\mathbf{q}, \mathbf{u}) dt = d\mathbf{R} + \mathbf{B}(\mathbf{q}) d\boldsymbol{\Gamma}. \quad (2.23)$$

This form of representation of the Euler-Lagrange equations has wider range of validity such that it is valid "everywhere" instead of "almost everywhere". For the force measure $d\mathbf{R}$ following decomposition is valid:

$$d\mathbf{R} = \mathbf{f} dt + \mathbf{F}' d\sigma, \quad (2.24)$$

such that $\mathbf{f}$ and $\mathbf{F}'$ represent Lebesgue-measurable forces and Borel-measurable forces, respectively. Similarly the differential measure of controls is decomposed as:

$$d\mathbf{\Gamma} = \boldsymbol{\tau} dt + \boldsymbol{\zeta}' d\sigma. \quad (2.25)$$

Here $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}'$ represent the Lebesgue-measurable controls and the Borel-measurable controls, respectively. The Radon-Nykodym derivative of $d\mathbf{\Gamma}$ with respect to $d\sigma$ is given by $\boldsymbol{\zeta}'$. The differential measure of generalised velocities is given by:

$$d\mathbf{u} = \dot{\mathbf{u}} dt + \boldsymbol{\chi}' d\sigma \quad (2.26)$$

Here $\boldsymbol{\chi}'$ is the Radon-Nykodym derivative of the differential measure of generalised velocities with respect to $d\sigma$. The substitution of (2.24) into (2.23) along with equation (2.26) reveals:

$$\begin{aligned} \mathbf{M}(\mathbf{q}) \dot{\mathbf{u}} dt + \mathbf{M}(\mathbf{q}) \boldsymbol{\chi}' d\sigma - \mathbf{h}(\mathbf{q}, \mathbf{u}) dt = \\ (\mathbf{f} + \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}) dt + (\mathbf{F}' + \mathbf{B}(\mathbf{q}) \boldsymbol{\zeta}') d\sigma. \end{aligned} \quad (2.27)$$

Equation (2.27) can be split into a Lebesgue and Borel part as given below:

$$\begin{aligned} \mathbf{M}(\mathbf{q}) \boldsymbol{\chi}' d\sigma &= (\mathbf{F}' + \mathbf{B}(\mathbf{q}) \boldsymbol{\zeta}') d\sigma, \\ \mathbf{M}(\mathbf{q}) \dot{\mathbf{u}} dt - \mathbf{h}(\mathbf{q}, \mathbf{u}) dt &= (\mathbf{f} + \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}) dt. \end{aligned}$$

An impact in mechanics is defined as a discontinuity in the generalised velocities of a Lagrangian system which is induced by impulsive forces. From the Borel part one obtains after evaluation of the Lebesgue-Stieltjes integral over atom of impact time following impact equation:

$$\mathbf{M}(\mathbf{q}(t_i)) (\mathbf{u}_i^+ - \mathbf{u}_i^-) = \mathbf{F}_i^+ - \mathbf{F}_i^- + \mathbf{B}(\mathbf{q}(t_i)) (\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-), \quad (2.28)$$

where t_i is an element of discontinuity points of the velocity $\mathbf{u}$. The difference given by $\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-$ is equal to the impulsive control $\mathbf{\Gamma}$ that acts on the Lagrangian system at an instant $t_i \in \mathcal{I}_T$. The difference given by $\mathbf{F}_i^+ - \mathbf{F}_i^-$ denotes the impulsive force interaction with the surroundings $\mathbf{R}$. The Lebesgue part is expressed in two equivalent forms as below:

$$\begin{aligned} \mathbf{M}(\mathbf{q}^+) \dot{\mathbf{u}}^+ dt - \mathbf{h}(\mathbf{q}^+, \mathbf{u}^+) dt &= (\mathbf{f}^+ + \mathbf{B}(\mathbf{q}^+) \boldsymbol{\tau}^+) dt, \\ \mathbf{M}(\mathbf{q}^-) \dot{\mathbf{u}}^- dt - \mathbf{h}(\mathbf{q}^-, \mathbf{u}^-) dt &= (\mathbf{f}^- + \mathbf{B}(\mathbf{q}^-) \boldsymbol{\tau}^-) dt. \end{aligned}$$

The points of discontinuity are Lebesgue negligible. As a corollary, the directional Euler-Lagrange equations can be stated as follows:

$$\mathbf{M}(\mathbf{q}) \dot{\mathbf{u}}^+ - \mathbf{h}(\mathbf{q}, \mathbf{u}^+) = \mathbf{f}^+ + \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}^+, \quad (2.29)$$

$$\mathbf{M}(\mathbf{q}) \dot{\mathbf{u}}^- - \mathbf{h}(\mathbf{q}, \mathbf{u}^-) = \mathbf{f}^- + \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}^-. \quad (2.30)$$

2.3 The Measure Differential Inclusion of Motion for Finite-Dimensional Lagrangian Systems with Controls

In this section, the set-valued force laws that emanate from the interaction of the mechanical system with its surroundings is expounded. The relation to the extended principle of Gauss is stressed in which the controls appear explicitly. Two forms of the Quadratic Program (QP), that emanate from the extended Gauss Principle concept are introduced. The first one is the form that represents the formulation of the dual extend principle of Gauss in impact free motion. The second one includes a QP in impulsive forces and how impulsive controls come into play.

Set-valued force laws are characterised by their conditional dependence on geometrical and kinematical entities, because of this dependence this relation between forces and kinematic entities are investigated in detail. The unilateral forces, which are nonpotential in the classical sense, are incorporated by the appropriate generalised force directions in the generalised force vector $\mathbf{f}$ as described in equation (2.21). The controls is introduced into the equations of motion by means of the structure of $\mathbf{f}$. In order to formulate the contact situations on position, velocity and acceleration levels properly following index sets are defined:

$$\mathcal{I}_G = \{1, 2, \dots, z\}, \quad (2.31)$$

$$\mathcal{I}_S = \{i \in \mathcal{I}_G \mid g_{u_i} = 0\}, \quad (2.32)$$

$$\mathcal{I}_N = \{i \in \mathcal{I}_G \mid g_{u_i} = 0, \gamma_{u_i} = 0\}. \quad (2.33)$$

$\mathcal{I}_G$ denotes the set of all contacts that can occur on position level of the nonsmooth mechanical system and the total number amounts to z . $\mathcal{I}_S$ denotes the set of all contacts that are closed on position level of the system and the total number amounts to k . $\mathcal{I}_N$ denotes the set of all contacts for which normal contact velocity and normal contact distance equal to zero, and the number of elements amounts to m .

The tangential and normal local kinematics need to be defined in order to relate the contact distance to the set-valued force element. For the detection of the closing of a contact let the vector $\mathbf{g}_u(\mathbf{q})$ represent the normal contact distances between the rigid bodies in the system which are always nonnegative. The normal and tangential contact velocities γ_u and γ_s are defined as:

$$\gamma_u = \mathbf{W}_u^T(\mathbf{q}) \mathbf{u}, \quad (2.34)$$

$$\gamma_s = \mathbf{W}_s^T(\mathbf{q}) \mathbf{u}, \quad (2.35)$$

respectively, and γ_u is obtained as the total time derivative of $\mathbf{g}_u(\mathbf{q})$. The normal and tangential

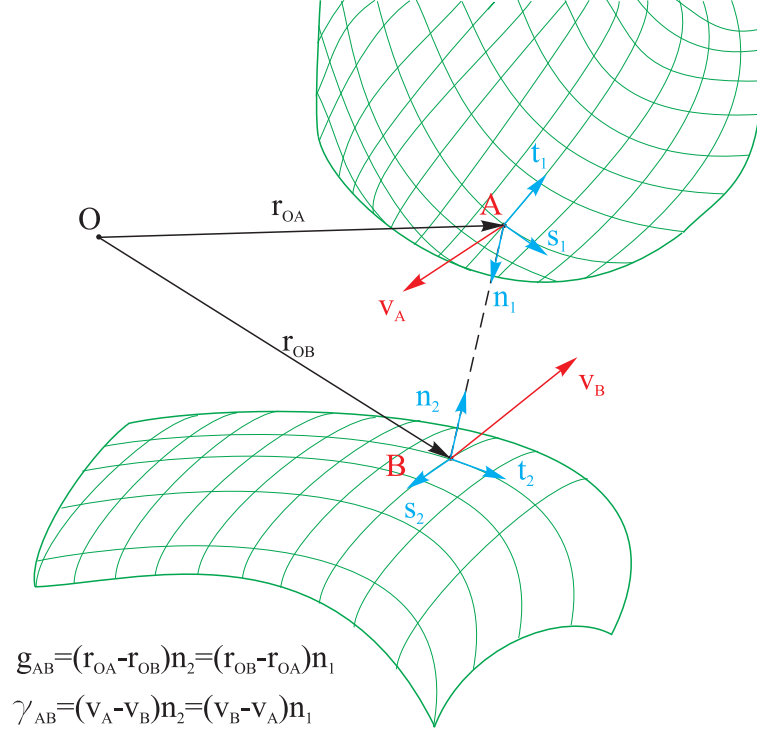


Figure 2.3: General contact kinematics.

contact accelerations are given by the following equations:

$$\dot{\gamma}_u = \mathbf{W}_u^T(\mathbf{q}) \dot{\mathbf{u}} + \boldsymbol{\omega}_u(\mathbf{q}, \mathbf{u}), \quad (2.36)$$

$$\dot{\gamma}_s = \mathbf{W}_s^T(\mathbf{q}) \dot{\mathbf{u}} + \boldsymbol{\omega}_s(\mathbf{q}, \mathbf{u}). \quad (2.37)$$

Further, the definition of following index sets are made on acceleration level:

$$\mathcal{C}_{u_i} = \{\lambda_{u_i} \mid \lambda_{u_i} \geq 0, \forall i \in \mathcal{I}_G\}, \quad (2.38)$$

$$\mathcal{C}_{s_i(\lambda_{u_i})} = \{\boldsymbol{\lambda}_{s_i} \mid |\boldsymbol{\lambda}_{s_i}| \leq \mu_i \lambda_{u_i}, \forall i \in \mathcal{I}_N\}, \quad (2.39)$$

where the vectors $\boldsymbol{\lambda}_{s_i}, \lambda_{u_i}$ are the tangential and the normal contact forces, and μ_i denotes the friction coefficient at contact i . For different friction models such as Coulomb-Contensou friction and details about set-valued friction analysis references [51] by Le Saux *et al.* and [52] by Leine *et al.* provide a good overview. In the framework of this thesis isotropic friction laws are considered, for the modeling of nonisotropic friction the reference [46] is recommended. The differential inclusion of a general nonautonomous mechanical system subject to spatial friction and unilateral contact forces in the absence of impacts is stated as:

$$\mathbf{M}(\mathbf{q}) \ddot{\mathbf{u}} - \mathbf{h}(\mathbf{q}, \mathbf{u}) - \mathbf{W}_s(\mathbf{q}) \boldsymbol{\lambda}_s - \mathbf{W}_u(\mathbf{q}) \boldsymbol{\lambda}_u - \mathbf{B}(\mathbf{q}) \boldsymbol{\tau} = \mathbf{0} \quad \text{a.e.}, \quad (2.40)$$

$$\dot{\gamma}_{u_i} \in \mathcal{N}_{\mathcal{C}_{u_i}}(\lambda_{u_i}), \quad \forall i \in \mathcal{I}_N, \quad (2.41)$$

$$-\dot{\gamma}_{s_i} \in \mathcal{N}_{\mathcal{C}_{s_i(\lambda_{u_i})}}(\boldsymbol{\lambda}_{s_i}), \quad \forall i \in \mathcal{I}_N. \quad (2.42)$$

The vector $\boldsymbol{\tau} \in \mathbb{R}^s$ denotes the vector of Lebesgue measurable control inputs. The normal cone representation in friction problems and contact mechanics are first treated in [6] by Alart *et al.* and [65] by Moreau *et al.*, respectively. If a mechanical system with n degrees of freedom with m contacts, that have spatial friction is considered, then $\mathbf{W}_s(\mathbf{q})$ is a $n \times 2m$ linear operator of generalised friction force directions; $\mathbf{W}_u(\mathbf{q})$ is a $n \times m$ linear operator of generalised normal force directions.

In order to treat impacts in this framework constitutive laws need to be introduced. In [45] by Glocker a representation of Moreau's impact law in local contact coordinates has been derived, showing that:

$$0 \leq \Lambda_i \perp (\gamma_i^+ + \epsilon \gamma_i^-) \geq 0, \quad 1 \geq \epsilon \geq 0. \quad (2.43)$$

for each contact $i \in \mathcal{I}_S$. These concepts can be extended to the tangential and normal impact by introducing following variables ξ_{s_i} and ξ_{u_i} , which vectorially are:

$$\boldsymbol{\xi}_s = \boldsymbol{\gamma}_s^+ + \epsilon_s \boldsymbol{\gamma}_s^-, \quad (2.44)$$

$$\boldsymbol{\xi}_u = \boldsymbol{\gamma}_u^+ + \epsilon_u \boldsymbol{\gamma}_u^-. \quad (2.45)$$

These slack variables of kinematics enable the blending of the extended Moreau-Newton impact law with the linear complementarity framework. The details of this modeling approach of impacts and other impact models can be found in [42] by Glocker. The dynamics of a mechanical system is formulated in the measure-differential inclusion form as follows:

$$\mathbf{M}(\mathbf{q}) d\mathbf{u} - \mathbf{h}(\mathbf{q}, \mathbf{u}) dt - \mathbf{W}_s(\mathbf{q}) d\boldsymbol{\Lambda}_s - \mathbf{W}_u(\mathbf{q}) d\boldsymbol{\Lambda}_u - \mathbf{B}(\mathbf{q}) d\boldsymbol{\Gamma} = \mathbf{0}, \quad (2.46)$$

$$\xi_{u_i} \in \mathcal{N}_{\mathcal{C}_{u_i}}(d\boldsymbol{\Lambda}_{u_i}), \quad \forall i \in \mathcal{I}_S, \quad (2.47)$$

$$-\xi_{s_i} \in \mathcal{N}_{\mathcal{C}_{s_i}}(d\boldsymbol{\Lambda}_{s_i}), \quad \forall i \in \mathcal{I}_S. \quad (2.48)$$

Here $d\boldsymbol{\Lambda}_s$ and $d\boldsymbol{\Lambda}_u$ are the differential measures of the tangential and normal contact forces, respectively. The differential measure of the generalised velocity is given by $d\mathbf{u}$ and is given in equation (2.26). The contact force differential measures and the differential measure of controls are given by:

$$d\boldsymbol{\Lambda}_s = \boldsymbol{\lambda}_s dt + \boldsymbol{\Lambda}'_s d\sigma,$$

$$d\boldsymbol{\Lambda}_u = \boldsymbol{\lambda}_u dt + \boldsymbol{\Lambda}'_u d\sigma,$$

$$d\boldsymbol{\Gamma} = \boldsymbol{\tau} dt + \boldsymbol{\zeta}' d\sigma.$$

2.3.1 The Linear Complementarity (LCP) Representation of Measure Differential Inclusions

If the normal cones in the force laws are finitely generated, then the determination of accelerations and forces can be represented in a linear complementarity form. This is the case, when the

line of action of the friction forces are known, which is not the case for spatial friction. In the sequel between two types of frictional contacts is distinguished, because of their differing roles in the LCP. The set $\mathcal{I}_{\text{NA}}$ denotes the set of contacts at which the normal force, that induces the friction force is known a priori, whereas $\mathcal{I}_{\text{NC}}$ denote the set of contacts, where the value of the normal contact force depends on the friction force value at a given moment. In [63] it has been shown that the determination of the accelerations of a mechanical system subject to unilateral constraints without friction can be represented as a primal and dual quadratic programming problem. Further, it is shown that a generalisation of the Gauss' variational principle is valid in the case of unilateral constraints without friction. In [44] by Glocker, it is shown that a quadratic programming problem can be obtained if Tresca type friction, for which the normal force is decoupled from the tangential force, exists and that the equations of motion along with the linear-complementarity conditions constitute necessary Karush-Kuhn-Tucker conditions of optimality for the quadratic programming problem. The definition of following index sets are made:

$$\mathcal{C}_{\text{u}_i} = \{\lambda_{\text{u}_i} \mid \lambda_{\text{u}_i} \geq 0, \quad \forall i \in \mathcal{I}_{\text{G}}\}, \quad (2.49)$$

$$\mathcal{C}_{\text{s}_i(\lambda_{\text{u}_i})} = \{\lambda_{\text{sc}_i} \mid |\lambda_{\text{sc}_i}| \leq \mu_i \lambda_{\text{u}_i}, \quad \forall i \in \mathcal{I}_{\text{NC}} \subset \mathcal{I}_{\text{N}}\}, \quad (2.50)$$

$$\mathcal{C}_{\text{a}_i} = \{\lambda_{\text{sa}_i} \mid |\lambda_{\text{sa}_i}| \leq a_{\text{u}_i}, \quad \forall i \in \mathcal{I}_{\text{NA}} \subset \mathcal{I}_{\text{N}}\}, \quad (2.51)$$

such that $\mathcal{I}_{\text{NA}} \cap \mathcal{I}_{\text{NC}} = \emptyset$, $\mathcal{I}_{\text{NA}} \cup \mathcal{I}_{\text{NC}} = \mathcal{I}_{\text{N}}$, $N(\mathcal{I}_{\text{NA}}) = w$, $N(\mathcal{I}_{\text{NC}}) = v$, $N(\mathcal{I}_{\text{N}}) = m$ and $N(\mathcal{I}_{\text{S}}) = k$. Here, the vectors λ_{sa} and λ_{sc} , denote tangential contact forces of Tresca-type and Coulomb-type, respectively. The entity a_{u_i} denotes the apriori known sliding contact force at contact i . Further, for both type frictional contacts between sticking and sliding contacts is distinguished, which gives rise to the definition of new index sets such that $\mathcal{I}_{\text{NA}} = {}_{\text{H}}\mathcal{I}_{\text{NA}} \cup {}_{\text{G}}\mathcal{I}_{\text{NA}}$ and $\mathcal{I}_{\text{NC}} = {}_{\text{H}}\mathcal{I}_{\text{NC}} \cup {}_{\text{G}}\mathcal{I}_{\text{NC}}$ are valid. The number of elements of these sets are related to each other by

$$N(\mathcal{I}_{\text{NA}}) = w = N({}_{\text{H}}\mathcal{I}_{\text{NA}}) + N({}_{\text{G}}\mathcal{I}_{\text{NA}}) = p + s, \quad (2.52)$$

$$N(\mathcal{I}_{\text{NC}}) = v = N({}_{\text{H}}\mathcal{I}_{\text{NC}}) + N({}_{\text{G}}\mathcal{I}_{\text{NC}}) = r + t. \quad (2.53)$$

Here subscript H refers to sticking and G refers to sliding. Based on this classification, the vector γ_{s} is decomposed as $\gamma_{\text{s}} = \begin{bmatrix} \gamma_{\text{sa}} & \gamma_{\text{sc}} \end{bmatrix}^{\text{T}}$ and the related relative contact accelerations are given by:

$$\dot{\gamma}_{\text{sa}} = \mathbf{W}_{\text{sa}}^{\text{T}} \dot{\mathbf{u}} + \omega_{\text{sa}}, \quad (2.54)$$

$$\dot{\gamma}_{\text{sc}} = \mathbf{W}_{\text{sc}}^{\text{T}} \dot{\mathbf{u}} + \omega_{\text{sc}}. \quad (2.55)$$

The force λ_{sc} can be decomposed into sliding contact forces ${}_{\text{G}}\lambda_{\text{sc}}$ and sticking contact forces ${}_{\text{H}}\lambda_{\text{sc}}$ at a given instant uniquely. The tangential contact forces and relative tangential contact velocities of contacts with Tresca-type friction can be decomposed analogously. The differential

inclusion of a general nonautonomous mechanical system subject to planar friction (known line of action) and unilateral contact forces is stated as:

$$\mathbf{M}(\mathbf{q}) \dot{\mathbf{u}} - \bar{\mathbf{h}}(\mathbf{q}, \mathbf{u}) - {}_H\mathbf{W}_{sc}(\mathbf{q}) {}_H\boldsymbol{\lambda}_{sc} - {}_H\mathbf{W}_{sa}(\mathbf{q}) {}_H\boldsymbol{\lambda}_{sa} - \mathbf{W}_u \boldsymbol{\lambda}_u - \mathbf{B}(\mathbf{q}) \boldsymbol{\tau} = \mathbf{0}, \quad (2.56)$$

$$\lambda_{u_i} \in \text{Upr}(\dot{\gamma}_{u_i}), \quad \forall i \in \mathcal{I}_N, \quad (2.57)$$

$$-\lambda_{sc_j} \in \mu_j \lambda_{u_j} \text{Sgn}(\dot{\gamma}_{sc_j}), \quad \forall j \in {}_H\mathcal{I}_{NC}, \quad (2.58)$$

$$-\lambda_{sa_k} \in a_{u_k} \text{Sgn}(\dot{\gamma}_{sa_k}), \quad \forall k \in {}_H\mathcal{I}_{NA}. \quad (2.59)$$

Defining $\bar{\mathbf{h}} = \mathbf{h} + {}_G\mathbf{W}_{sc} {}_G\boldsymbol{\lambda}_{sc} + {}_G\mathbf{W}_{sa} {}_G\boldsymbol{\lambda}_{sa}$, enables the incorporation of all sliding contact forces in the vector $\bar{\mathbf{h}}$. Here, the linear operator $\mathbf{W}_u$ has dimensions $n \times m$.

Given this setting, the linear complementarity that characterises the evolution of the Lagrangian system as described in relations (2.56) to (2.59) is derived in the sequel. The set-valued signum type friction force characteristics can be decomposed into two unilateral force laws by introducing new variables as depicted in figure 2.2 and given below:

$${}_H\dot{\gamma}_{sc} = {}_H\dot{\gamma}_{rc} - {}_H\dot{\gamma}_{lc}, \quad {}_H\dot{\gamma}_{sa} = {}_H\dot{\gamma}_{ra} - {}_H\dot{\gamma}_{la}, \quad (2.60)$$

$${}_H\boldsymbol{\lambda}_{lc} = \boldsymbol{\mu} {}_H\boldsymbol{\lambda}_u - {}_H\boldsymbol{\lambda}_{sc}, \quad {}_H\boldsymbol{\lambda}_{la} = {}_H\mathbf{a} - {}_H\boldsymbol{\lambda}_{sa}, \quad (2.61)$$

$${}_H\boldsymbol{\lambda}_{rc} = \boldsymbol{\mu} {}_H\boldsymbol{\lambda}_u + {}_H\boldsymbol{\lambda}_{sc}, \quad {}_H\boldsymbol{\lambda}_{ra} = {}_H\mathbf{a} + {}_H\boldsymbol{\lambda}_{sa}, \quad (2.62)$$

along with following nonnegativity and complementarity conditions:

$$\langle {}_H\dot{\gamma}_u, {}_H\boldsymbol{\lambda}_u \rangle = 0, \quad {}_H\dot{\gamma}_u \succeq \mathbf{0}, \quad {}_H\boldsymbol{\lambda}_u \succeq \mathbf{0}, \quad (2.63)$$

$$\langle {}_H\dot{\gamma}_{rc}, {}_H\boldsymbol{\lambda}_{rc} \rangle = 0, \quad {}_H\dot{\gamma}_{rc} \succeq \mathbf{0}, \quad {}_H\boldsymbol{\lambda}_{rc} \succeq \mathbf{0}, \quad (2.64)$$

$$\langle {}_H\dot{\gamma}_{lc}, {}_H\boldsymbol{\lambda}_{lc} \rangle = 0, \quad {}_H\dot{\gamma}_{lc} \succeq \mathbf{0}, \quad {}_H\boldsymbol{\lambda}_{lc} \succeq \mathbf{0}, \quad (2.65)$$

$$\langle {}_H\dot{\gamma}_{ra}, {}_H\boldsymbol{\lambda}_{ra} \rangle = 0, \quad {}_H\dot{\gamma}_{ra} \succeq \mathbf{0}, \quad {}_H\boldsymbol{\lambda}_{ra} \succeq \mathbf{0}, \quad (2.66)$$

$$\langle {}_H\dot{\gamma}_{la}, {}_H\boldsymbol{\lambda}_{la} \rangle = 0, \quad {}_H\dot{\gamma}_{la} \succeq \mathbf{0}, \quad {}_H\boldsymbol{\lambda}_{la} \succeq \mathbf{0}. \quad (2.67)$$

Here $\boldsymbol{\mu} \in \mathbb{R}^{r \times r}$ is a diagonal matrix with friction coefficients. The vector $\mathbf{a} \in \mathbb{R}^p$ denotes the normal force vector for frictional contacts of Tresca-type. Further ${}_H\boldsymbol{\lambda}_{rc}, {}_H\boldsymbol{\lambda}_{lc} \in \mathbb{R}^r$; ${}_H\boldsymbol{\lambda}_{ra}, {}_H\boldsymbol{\lambda}_{la} \in \mathbb{R}^p$; ${}_H\boldsymbol{\gamma}_{rc}, {}_H\boldsymbol{\gamma}_{lc} \in \mathbb{R}^r$ and ${}_H\boldsymbol{\gamma}_{ra}, {}_H\boldsymbol{\gamma}_{la} \in \mathbb{R}^p$ and are related entities to sticking contacts. The generalised accelerations of the system are given by:

$$\dot{\mathbf{u}} = \mathbf{M}^{-1}(\mathbf{q}) (\bar{\mathbf{h}}(\mathbf{q}, \mathbf{u}) + {}_H\mathbf{W}_{sc}(\mathbf{q}) {}_H\boldsymbol{\lambda}_{sc} + {}_H\mathbf{W}_{sa}(\mathbf{q}) {}_H\boldsymbol{\lambda}_{sa} + {}_H\mathbf{W}_u(\mathbf{q}) {}_H\boldsymbol{\lambda}_u + \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}). \quad (2.68)$$

Insertion of (2.68) in the expressions for relative contact accelerations given in equations (2.36), (2.54) and (2.55) reveals following set of equations:

$${}_H\dot{\gamma}_u = {}_H\mathbf{W}_u^T \mathbf{M}^{-1} (\bar{\mathbf{h}} + {}_H\mathbf{W}_{sc} {}_H\boldsymbol{\lambda}_{sc} + {}_H\mathbf{W}_{sa} {}_H\boldsymbol{\lambda}_{sa} + {}_H\mathbf{W}_u {}_H\boldsymbol{\lambda}_u + \mathbf{B} \boldsymbol{\tau}) + {}_H\boldsymbol{\omega}_u, \quad (2.69)$$

$${}_H\dot{\gamma}_{sc} = {}_H\mathbf{W}_{sc}^T \mathbf{M}^{-1} (\bar{\mathbf{h}} + {}_H\mathbf{W}_{sc} {}_H\boldsymbol{\lambda}_{sc} + {}_H\mathbf{W}_{sa} {}_H\boldsymbol{\lambda}_{sa} + {}_H\mathbf{W}_u {}_H\boldsymbol{\lambda}_u + \mathbf{B} \boldsymbol{\tau}) + {}_H\boldsymbol{\omega}_{sc}, \quad (2.70)$$

$${}_H\dot{\gamma}_{sa} = {}_H\mathbf{W}_{sa}^T \mathbf{M}^{-1} (\bar{\mathbf{h}} + {}_H\mathbf{W}_{sc} {}_H\boldsymbol{\lambda}_{sc} + {}_H\mathbf{W}_{sa} {}_H\boldsymbol{\lambda}_{sa} + {}_H\mathbf{W}_u {}_H\boldsymbol{\lambda}_u + \mathbf{B} \boldsymbol{\tau}) + {}_H\boldsymbol{\omega}_{sa}, \quad (2.71)$$

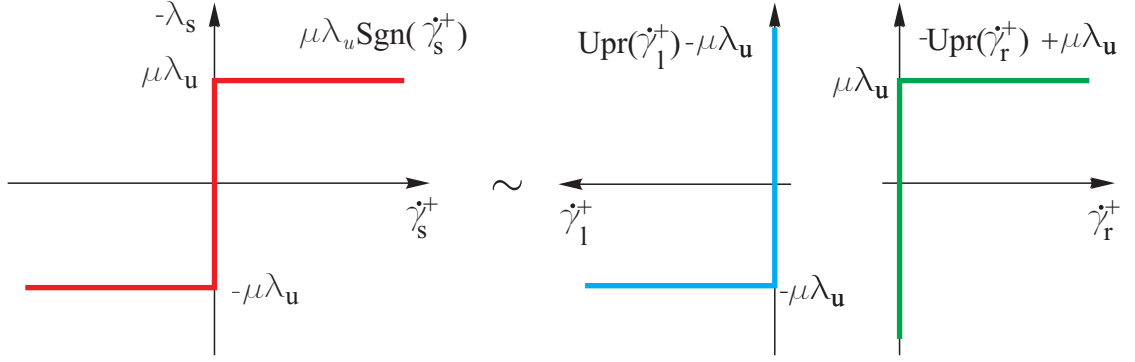


Figure 2.4: Decomposition of the set-valued Sgn relation into two Unilateral Primitives.

which are arranged in the form of a linear complementarity problem:

$$\mathbf{y} = \mathbf{A} \mathbf{x} + \mathbf{b}, \quad (2.72)$$

$$\mathbf{y} \succeq \mathbf{0}, \quad \mathbf{x} \succeq \mathbf{0}, \quad \langle \mathbf{x}, \mathbf{y} \rangle = 0. \quad (2.73)$$

Where the complementarity vectors $\mathbf{x}$ and $\mathbf{y}$ are identified as:

$$\mathbf{x} = \begin{bmatrix} {}_H\boldsymbol{\lambda}_u^T & {}_H\boldsymbol{\lambda}_{rc}^T & {}_H\boldsymbol{\lambda}_{ra}^T & {}_H\dot{\gamma}_{lc}^T & {}_H\dot{\gamma}_{la}^T \end{bmatrix}^T, \quad (2.74)$$

$$\mathbf{y} = \begin{bmatrix} {}_H\dot{\gamma}_u^T & {}_H\dot{\gamma}_{rc}^T & {}_H\dot{\gamma}_{ra}^T & {}_H\boldsymbol{\lambda}_{lc}^T & {}_H\boldsymbol{\lambda}_{la}^T \end{bmatrix}^T, \quad (2.75)$$

respectively. In the sequel for ease of notation the entities ${}_H\mathbf{W}_{sc} \in \mathbb{R}^{n \times r}$, ${}_H\boldsymbol{\lambda}_{sc} \in \mathbb{R}^r$, ${}_H\mathbf{W}_{sa} \in \mathbb{R}^{n \times p}$, ${}_H\boldsymbol{\lambda}_{sa} \in \mathbb{R}^p$ is represented by $\mathbf{W}_{sc}$, $\boldsymbol{\lambda}_{sc}$, $\mathbf{W}_{sa}$, $\boldsymbol{\lambda}_{sa}$, respectively. The linear operators $\mathbf{A} \in \mathbb{R}^{(m+2p+2r) \times (m+2p+2r)}$ and vector $\mathbf{b} \in \mathbb{R}^{(m+2p+2r)}$ are given by:

$$\mathbf{A} = \begin{bmatrix} \mathbf{W}_u^T \mathbf{M}^{-1} (\mathbf{W}_u - \mathbf{W}_{sc} \boldsymbol{\mu}) & \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{W}_{sc} & \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{W}_{sa} & \mathbf{0}_{m \times r} & \mathbf{0}_{m \times p} \\ \mathbf{W}_{sc}^T \mathbf{M}^{-1} (\mathbf{W}_u - \mathbf{W}_{sc} \boldsymbol{\mu}) & \mathbf{W}_{sc}^T \mathbf{M}^{-1} \mathbf{W}_{sc} & \mathbf{W}_{sc}^T \mathbf{M}^{-1} \mathbf{W}_{sa} & \mathbf{I}_{r \times r} & \mathbf{0}_{r \times p} \\ \mathbf{W}_{sa}^T \mathbf{M}^{-1} (\mathbf{W}_u - \mathbf{W}_{sc} \boldsymbol{\mu}) & \mathbf{W}_{sa}^T \mathbf{M}^{-1} \mathbf{W}_{sc} & \mathbf{W}_{sa}^T \mathbf{M}^{-1} \mathbf{W}_{sa} & \mathbf{0}_{p \times r} & \mathbf{I}_{p \times p} \\ 2\boldsymbol{\mu} & -\mathbf{I}_{r \times r} & \mathbf{0}_{r \times p} & \mathbf{0}_{r \times r} & \mathbf{0}_{r \times p} \\ \mathbf{0}_{p \times m} & \mathbf{0}_{p \times r} & -\mathbf{I}_{p \times p} & \mathbf{0}_{p \times r} & \mathbf{0}_{p \times p} \end{bmatrix}, \quad (2.76)$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{W}_u^T \mathbf{f}_{\dot{u}} + \boldsymbol{\omega}_u & \mathbf{W}_{sc}^T \mathbf{f}_{\dot{u}} + \boldsymbol{\omega}_{sc} & \mathbf{W}_{sa}^T \mathbf{f}_{\dot{u}} + \boldsymbol{\omega}_{sa} & \mathbf{0}_{r \times 1} & 2\mathbf{a} \end{bmatrix}^T, \quad (2.77)$$

respectively, where $\mathbf{f}_{\dot{u}}$ is given by $\mathbf{f}_{\dot{u}} = \mathbf{M}^{-1}(\bar{\mathbf{h}} + \mathbf{B} \boldsymbol{\tau})$.

Defining

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_u \\ \cdots \\ \gamma_{ra} \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_u \\ \cdots \\ \lambda_{ra} \end{bmatrix}, \quad (2.78)$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_u \\ \cdots \\ \omega_{ra} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{W}_u & \vdots & \mathbf{W}_{sa} \end{bmatrix}. \quad (2.79)$$

The quadratic programming problem, which is the dual problem to the extended principle of Gauss, is given by:

$$\min_{\lambda, \tau} f(\lambda, \tau) = \frac{1}{2} \langle \lambda, \mathbf{G} \lambda \rangle + \langle \mathbf{g}, \lambda \rangle + \langle \mathbf{l}, \tau \rangle \quad (2.80)$$

subject to inequalities:

$$\lambda_u \succeq \mathbf{0}, \quad \lambda_{ra} \succeq \mathbf{0}, \quad 2\mathbf{a} - \lambda_{ra} \succeq \mathbf{0}. \quad (2.81)$$

Various terms in (2.80) are given by:

$$\mathbf{G} = \mathbf{W}^T \mathbf{M}^{-1} \mathbf{W}, \quad (2.82)$$

$$\mathbf{g} = \mathbf{W}^T \mathbf{M}^{-1} (-\mathbf{W} \mathbf{a} + \mathbf{h}) + \omega, \quad (2.83)$$

$$\mathbf{l} = -\mathbf{W}^T \mathbf{M}^{-1} \mathbf{W} \mathbf{B}. \quad (2.84)$$

There are several aspects of the optimisation problem:

- If at no contact Coulomb type friction exists then the dual optimisation problem exists.
- The solution to the problem in (2.80) is unique if and only if $\mathbf{G}$ is positive definite, which is only the case if the matrix of generalised contact force directions $\mathbf{W}$ has full rank. (Since $\mathbf{M}^{-1}$ is always PD).

2.3.2 Impact Equation

Modeling of impacts in the realm of rigid-body multi-body systems is an ongoing field of research. In the sequel geometric and kinematic consistency of collision and impact situations is assumed. In order to derive the impact equation one has to revert to the decomposition of the equations of motion, in its Lebesgue and Borel measurable parts. Given this decomposition of the differential force measures the directional Euler-Lagrange equations can be expressed as:

$$\mathbf{M}(\mathbf{q}) \dot{\chi}' d\sigma - \mathbf{W}_{sc}(\mathbf{q}) \Lambda'_{sc} d\sigma - \mathbf{W}_{sa}(\mathbf{q}) \Lambda'_{sa} d\sigma - \mathbf{W}_u(\mathbf{q}) \Lambda'_u d\sigma - \mathbf{B}(\mathbf{q}) \zeta' d\sigma = \mathbf{0}, \quad (2.85)$$

$$\mathbf{M}(\mathbf{q}) \dot{\mathbf{u}} dt - \mathbf{h}(\mathbf{q}, \mathbf{u}) dt - \mathbf{W}_{sc}(\mathbf{q}) \lambda_{sc} dt - \mathbf{W}_{sa}(\mathbf{q}) \lambda_{sa} dt - \mathbf{W}_u(\mathbf{q}) \lambda_u dt - \mathbf{B}(\mathbf{q}) \tau dt = \mathbf{0}. \quad (2.86)$$

The contact force differential measures are given by:

$$d\Lambda_{sc} = \lambda_{sc} dt + \Lambda'_{sc} d\sigma,$$

$$d\Lambda_{sa} = \lambda_{sa} dt + \Lambda'_{sa} d\sigma,$$

$$d\Lambda_u = \lambda_u dt + \Lambda'_u d\sigma,$$

$$d\Gamma = \tau dt + \zeta' d\sigma.$$

By Lebesgue-Stieltjes integration of the Borel part given in equation (2.86) over the atom of impact time the impact equation is obtained:

$$\int_{\{t_{\text{imp}}\}} \mathbf{M} \boldsymbol{\chi}' d\sigma - \mathbf{W}_{\text{sc}} \boldsymbol{\Lambda}'_{\text{sc}} d\sigma - \mathbf{W}_{\text{sa}} \boldsymbol{\Lambda}'_{\text{sa}} d\sigma - \mathbf{W}_{\text{u}} \boldsymbol{\Lambda}'_{\text{u}} d\sigma - \mathbf{B} \boldsymbol{\zeta}' d\sigma = \quad (2.87)$$

$$\mathbf{M} (\boldsymbol{\chi}^+ - \boldsymbol{\chi}^-) - \mathbf{W}_{\text{sc}} (\boldsymbol{\Lambda}_{\text{sc}}^+ - \boldsymbol{\Lambda}_{\text{sc}}^-) - \mathbf{W}_{\text{u}} (\boldsymbol{\Lambda}_{\text{u}}^+ - \boldsymbol{\Lambda}_{\text{u}}^-) - \mathbf{B} (\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-). \quad (2.88)$$

Following expressions are substituted:

$$\begin{aligned} \mathbf{u}^+ - \mathbf{u}^- &= \boldsymbol{\chi}^+ - \boldsymbol{\chi}^-, \\ \boldsymbol{\Lambda}_{\text{sc}} &= \boldsymbol{\Lambda}_{\text{sc}}^+ - \boldsymbol{\Lambda}_{\text{sc}}^-, \\ \boldsymbol{\Lambda}_{\text{u}} &= \boldsymbol{\Lambda}_{\text{u}}^+ - \boldsymbol{\Lambda}_{\text{u}}^-, \\ \boldsymbol{\Gamma} &= \boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-. \end{aligned}$$

Here $\boldsymbol{\Lambda}_{\text{sc}_i}$ is the tangential contact impact impulse at a contact for $i \in \mathcal{I}_{\text{sc}}$, $\boldsymbol{\Lambda}_{\text{u}_i}$ is the normal contact impact impulse for $i \in \mathcal{I}_{\text{S}}$ and $\boldsymbol{\Gamma}$ is the impulsive control action. A superscripted $+$ denotes the post-impact value of the related entity, and analogously superscripted $-$ denotes the pre-impact value of the related entity. The closing of a contact is not sufficient for an impact to occur. The closing of a contact may or may not be accompanied by an impact. In order to formulate the occurrence of an impact conditionally following entity is defined:

$$\boldsymbol{\xi}_{\text{u}} = \boldsymbol{\gamma}_{\text{u}}^+ + \boldsymbol{\epsilon}_{\text{u}} \boldsymbol{\gamma}_{\text{u}}^-, \quad 1 \geq \boldsymbol{\epsilon}_{\text{u}} \geq 0 \quad (2.89)$$

so that the extended Moreau-Newton impact law is formulated as follows:

$$\boldsymbol{\xi}_{\text{u}} \succeq \mathbf{0}, \quad \boldsymbol{\Lambda}_{\text{u}} = \mathbf{0}, \quad (2.90)$$

$$\boldsymbol{\xi}_{\text{u}} = \mathbf{0}, \quad \boldsymbol{\Lambda}_{\text{u}} \succeq \mathbf{0}. \quad (2.91)$$

This relation is illustrated in figure 2.5. The interaction between frictional impact and normal impact needs to be established. If the normal cones given in equations are finitely generated, which is the case in one-dimensional friction a linear complementarity problem can be set-up. By "one-dimensional" is a pre-specified line of action of the friction force implied. This idea of prespecified line of action also applies in tangential impact considerations. If impacts are going to be incorporated a constitutive impact law is needed. The Newton impact law in normal direction can be stated as follows in the case of an impact:

$$\boldsymbol{\gamma}_{\text{u}}^+ = -\boldsymbol{\epsilon}_{\text{u}} \boldsymbol{\gamma}_{\text{u}}^-, \quad (2.92)$$

where $\boldsymbol{\epsilon}_{\text{u}} \in \mathbb{R}^{m \times m}$ is a diagonal matrix that entails the normal restitution coefficients. Considering that the tangential impulsive force is induced by the frictional involvement of the normal contact impulsive force, it can be stated as:

$$\boldsymbol{\Lambda}_{\text{u}_i} \in \text{Up}(\boldsymbol{\xi}_{\text{u}_i}), \quad (2.93)$$

$$-\boldsymbol{\Lambda}_{\text{s}_i} \in \mu_i \boldsymbol{\Lambda}_{\text{u}_i} \text{Sgn}(\boldsymbol{\xi}_{\text{s}_i}). \quad (2.94)$$

The tangential impact at a contact i can be reformulated in a similar way, noting that the impulse can be directed in both directions:

$$\xi_{s_i} = \gamma_{s_i}^+ + \epsilon_s \gamma_{s_i}^-. \quad (2.95)$$

By using the Newton-Moreau impact law the impact equations read:

$$\mathbf{M}(\mathbf{q}) (\mathbf{u}^+ - \mathbf{u}^-) - \mathbf{W}_{sc}(\mathbf{q}) \boldsymbol{\Lambda}_{sc} - \mathbf{W}_{sa}(\mathbf{q}) \boldsymbol{\Lambda}_{sa} - \mathbf{W}_u(\mathbf{q}) \boldsymbol{\Lambda}_u - \mathbf{B}(\mathbf{q}) \boldsymbol{\Gamma} = \mathbf{0}, \quad (2.96)$$

$$-\Lambda_{sc_i} \in \mu_i \text{Sgn}(\xi_{sc_i}), \quad \forall i \in \mathcal{I}_S \quad (2.97)$$

$$\Lambda_{u_i} \in \text{Upr}(\xi_{u_i}), \quad \forall i \in \mathcal{I}_S. \quad (2.98)$$

Defining the entities $\boldsymbol{\xi}_r, \boldsymbol{\xi}_l, \boldsymbol{\Lambda}_l, \boldsymbol{\Lambda}_r$ in order to decompose the impulse $\boldsymbol{\Lambda}_s$ and the slack variable $\boldsymbol{\xi}_s$ in left and right entities enables the fitting into the complementarity framework:

$$\boldsymbol{\xi}_s = \boldsymbol{\xi}_r - \boldsymbol{\xi}_l, \quad (2.99)$$

$$\boldsymbol{\Lambda}_l = \boldsymbol{\mu} \boldsymbol{\Lambda}_u - \boldsymbol{\Lambda}_s, \quad (2.100)$$

$$\boldsymbol{\Lambda}_r = \boldsymbol{\mu} \boldsymbol{\Lambda}_u + \boldsymbol{\Lambda}_s, \quad (2.101)$$

along with following complementarity conditions:

$$\boldsymbol{\xi}_u \succeq \mathbf{0}, \quad \boldsymbol{\Lambda}_u \succeq \mathbf{0}, \quad \langle \boldsymbol{\xi}_u, \boldsymbol{\Lambda}_u \rangle = 0, \quad (2.102)$$

$$\boldsymbol{\xi}_r \succeq \mathbf{0}, \quad \boldsymbol{\Lambda}_r \succeq \mathbf{0}, \quad \langle \boldsymbol{\xi}_r, \boldsymbol{\Lambda}_r \rangle = 0, \quad (2.103)$$

$$\boldsymbol{\xi}_l \succeq \mathbf{0}, \quad \boldsymbol{\Lambda}_l \succeq \mathbf{0}, \quad \langle \boldsymbol{\xi}_l, \boldsymbol{\Lambda}_l \rangle = 0. \quad (2.104)$$

Here $\boldsymbol{\mu} \in \mathbb{R}^{m \times m}$ denotes the diagonal matrix that has the contact friction coefficients as entries. The separation of the signum relation into two Upr relations is depicted in figure 2.6.

Insertion of equations (2.99) to (2.101) in equation (2.96) and solving for $\mathbf{u}^+$ results in:

$$\mathbf{u}^+ = \mathbf{u}^- + \mathbf{M}^{-1}(\mathbf{W}_u - \mathbf{W}_{sc}\boldsymbol{\mu}) \boldsymbol{\Lambda}_u + \mathbf{M}^{-1} \mathbf{W}_{sc} \boldsymbol{\Lambda}_r + \mathbf{M}^{-1} \mathbf{B} \boldsymbol{\Gamma}. \quad (2.105)$$

By using the equation (2.105), $\mathbf{u}^+$ can be eliminated from equations for post-impact relative velocities in normal and tangential direction:

$$\boldsymbol{\gamma}_u^+ = \mathbf{W}_u^T \mathbf{u}^- + \mathbf{W}_u^T \mathbf{M}^{-1}(\mathbf{W}_u - \mathbf{W}_{sc}\boldsymbol{\mu}) \boldsymbol{\Lambda}_u + \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{W}_{sc} \boldsymbol{\Lambda}_r + \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{B} \boldsymbol{\Gamma}, \quad (2.106)$$

$$\boldsymbol{\gamma}_{sc}^+ = \mathbf{W}_{sc}^T \mathbf{u}^- + \mathbf{W}_{sc}^T \mathbf{M}^{-1}(\mathbf{W}_u - \mathbf{W}_{sc}\boldsymbol{\mu}) \boldsymbol{\Lambda}_u + \mathbf{W}_{sc}^T \mathbf{M}^{-1} \mathbf{W}_{sc} \boldsymbol{\Lambda}_r + \mathbf{W}_{sc}^T \mathbf{M}^{-1} \mathbf{B} \boldsymbol{\Gamma}. \quad (2.107)$$

Insertion of equation (2.106) into equation (2.89) yields $\boldsymbol{\xi}_u$ in the following form:

$$\boldsymbol{\xi}_u = (\mathbf{I} + \boldsymbol{\epsilon}_u) \mathbf{W}_u^T \mathbf{u}^- + \mathbf{W}_u^T \mathbf{M}^{-1}(\mathbf{W}_u - \mathbf{W}_{sc}\boldsymbol{\mu}) \boldsymbol{\Lambda}_u + \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{W}_{sc} \boldsymbol{\Lambda}_r, \quad (2.108)$$

where $\mathbf{I}$ is an identity matrix of appropriate size. Combining results in the following expression for ξ_s :

$$\xi_{sc} = \xi_r - \xi_l = (\mathbf{I} + \epsilon_{sc}) \mathbf{W}_{sc}^T \mathbf{u}^- + \mathbf{W}_{sc}^T \mathbf{M}^{-1} (\mathbf{W}_u - \mathbf{W}_{sc} \mu) \Lambda_u + \mathbf{W}_{sc}^T \mathbf{M}^{-1} \mathbf{W}_{sc} \Lambda_r, \quad (2.109)$$

which can be arranged in the form of a linear complementarity problem:

$$\mathbf{y} = \mathbf{A} \mathbf{x} + \mathbf{b}, \quad (2.110)$$

$$\mathbf{y} \succeq \mathbf{0}, \quad \mathbf{x} \succeq \mathbf{0}, \quad \langle \mathbf{x}, \mathbf{y} \rangle = 0. \quad (2.111)$$

where the complementarity vector $\mathbf{x} \in \mathbb{R}^{2m}$ and $\mathbf{y} \in \mathbb{R}^{2,m}$ are stated as:

$$\mathbf{y} = \begin{bmatrix} \xi_u^T & \xi_r^T & \Lambda_l^T \end{bmatrix}^T, \quad \mathbf{x} = \begin{bmatrix} \Lambda_u^T & \Lambda_r^T & \xi_l^T \end{bmatrix}^T, \quad (2.112)$$

respectively. The matrix representation of the equations (2.106), (2.107), (2.108) and (2.109) along with the complementarity conditions given in equations (2.102), (2.103) and (2.104) represent the linear complementarity problem for a mechanical system with Newton impact and associated Coulomb friction. The matrix $\mathbf{A}$ and $\mathbf{b}$ are defined as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{W}_u^T \mathbf{M}^{-1} (\mathbf{W}_u - \mathbf{W}_{sc} \mu) & \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{W}_{sc} & \mathbf{0} \\ \mathbf{W}_{sc}^T \mathbf{M}^{-1} (\mathbf{W}_u - \mathbf{W}_{sc} \mu) & \mathbf{W}_{sc}^T \mathbf{M}^{-1} \mathbf{W}_{sc} & \mathbf{I} \\ 2\mu & -\mathbf{I} & \mathbf{0} \end{bmatrix}, \quad (2.113)$$

$$\mathbf{b} = \begin{bmatrix} (\mathbf{I} + \epsilon_u) \mathbf{W}_u^T \mathbf{u}^- + \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{B} \Gamma \\ \mathbf{I} + \epsilon_{sc} \mathbf{W}_{sc}^T \mathbf{u}^- + \mathbf{W}_{sc}^T \mathbf{M}^{-1} \mathbf{B} \Gamma \\ \mathbf{0} \end{bmatrix}. \quad (2.114)$$

By making use of structural properties of the LCP($\mathbf{A}, \mathbf{b}$) a quadratic programming problem can be defined. The quadratic programming problem is given by:

$$\min_{\Lambda_u, \Gamma} f(\Lambda_u, \Gamma) = \frac{1}{2} \langle \Lambda_u, \mathbf{G} \Lambda_u \rangle + \langle \mathbf{g}, \Lambda_u \rangle + \langle \mathbf{l}, \Gamma \rangle, \quad (2.115)$$

subject to:

$$\Lambda_u \succeq \mathbf{0} \quad (2.116)$$

which is the dual problem to the extended principle of Gauss in impact form with impulsive controls. Several linear operators are defined as follows:

$$\mathbf{G} = \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{W}_u, \quad (2.117)$$

$$\mathbf{g} = (\mathbf{I} + \epsilon_u) \mathbf{W}_u^T \mathbf{u}^-, \quad (2.118)$$

$$\mathbf{l} = \mathbf{W}_u^T \mathbf{M}^{-1} \mathbf{B}. \quad (2.119)$$

There are several aspects of the optimisation problem:

- In stating the optimisation problem for extended principle of Gauss in impactive form and impulsive control form, it is assumed that kinematic and geometric compatibility is given.
- The solution to the problem in (2.80) is unique if and only if $\mathbf{G}$ is positive definite, which is only the case if the matrix of generalised contact force directions $\mathbf{W}_u$ has full rank. (Since M^{-1} is always PD).
- Except kinematic and geometric compatibility, the existence of QP is dependent on whether the linear operator $\mathbf{A}$ is bisymmetrical or not. The bisymmetry of the linear operator is maintained if all μ_i 's in (2.94) are zero. This means that all impactive interactions in the tangential contact direction are decoupled from the normal contact interaction.

2.4 Modeling Underactuated Manipulators with Impactively Blockable Degrees of Freedom

The physical realisation of impactive blocking is achieved in several ways, however its modeling can be seen as a Coulomb like friction force characteristics with controlled/adjustable height of the set-valued signum relation. In order for the blocking action to take place, the height of the set-valued signum relation must rise high enough to reduce the relative velocity to zero immediately and must be zero if unblocked. This relation between a bilateral perfect constraint and free direction of motion is shown graphically in figures 2.5 and 2.7. In the sequel a class of mechanical systems are modeled and investigated, in which the only impulsive action is generated by the control action and it is assumed that there are no contact interactions with the surroundings. This class of mechanical systems have blockable DOF, which is described by an impulsive set-valued control law. Set-valued force elements are used in order to derive the complementarity structure of the blocking control. The relevance of unilateral analysis in mechanics is therefore closely related to the set-valued impulsive control of systems with blockable DOF. Set-valued force laws in mechanics are treated systematically in [44] by Glocker. In references [62] and [61] by Möller *et al.* the extension of the set-valued force law analysis to basic elements in electrical circuits can be found. The result of this subsection as presented in equations (2.166) to (2.172) is already stated in [115] and [116] by Yunt *et al.* as a result and used in the NLP methods. A practical introduction to the concept of MDI in the framework of impulsive control is provided in [110] by Yunt.

The dynamics of a mechanical system with n degrees of freedom, p blockable directions of motion and s Lebesgue measurable controls is formulated as a measure-differential inclusion as

follows:

$$\mathbf{M}(\mathbf{q}(t)) d\mathbf{u} - \mathbf{h}(\mathbf{q}(t), \mathbf{u}(t^+)) dt - \mathbf{B}(\mathbf{q}(t)) d\mathbf{\Gamma} = \mathbf{0}, \quad (2.120)$$

$$-d\Gamma_{b_i} \in dN_i \text{Sgn}(\gamma_{b_i}^+), \quad \forall i \in \mathcal{I}_B, \quad (2.121)$$

$$|\gamma_{b_i}^+| \in \text{Upr}(dN_i), \quad \forall i \in \mathcal{I}_B, \quad (2.122)$$

$$\boldsymbol{\tau} \in \mathcal{C}_\tau. \quad (2.123)$$

Here $\mathbf{q}(t)$ and $\mathbf{u}(t^+)$ denote the absolutely continuous ($\mathcal{AC}$) generalised positions and right continuous locally bounded variation ($\mathcal{RCLBV}$) generalised velocities, respectively. The set of all blockable DOF is denoted by $\mathcal{I}_B$. The linear operator $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n \times n}$ is the mass matrix, $\mathbf{h}(\mathbf{q}, \mathbf{u}) \in \mathbb{R}^{n \times 1}$ denote the vector of gyroscopical, centripetal and coriolis, smooth potential (gravity, spring etc.) forces and $d\mathbf{\Gamma} \in \mathbb{R}^{s \times 1}$ is the differential measure of controls which can be unbounded, set-valued and impulsive. The set $\mathcal{C}_\tau$ denotes the bounds on the Lebesgue-measurable single-valued ordinary controls, and is assumed to be of box-constrained type. The matrix $\mathbf{B}(\mathbf{q}) \in \mathbb{R}^{n \times s}$ includes the generalised control directions. This representation is the maximal representation, meaning that all joints are unblocked. The relative joint velocity at any blockable direction is a linear combination of the generalised velocities for scleronomic mechanical systems and is given by (2.124):

$$\gamma_{b_i} = \mathbf{w}_{b_i}(\mathbf{q}) \mathbf{u}, \quad \forall i \in \mathcal{I}_B. \quad (2.124)$$

For such a Lagrangian system a n -dimensional vector $\boldsymbol{\gamma} \in \mathbb{R}^n$ is defined:

$$\boldsymbol{\gamma} = \mathbf{W}^T(\mathbf{q}) \mathbf{u}, \quad (2.125)$$

where $\boldsymbol{\gamma} \in \mathbb{R}^p$ such that $\mathbf{w}_{b_i}(\mathbf{q}) \in \text{col}\{\mathbf{W}\}, \forall i \in \mathcal{I}_B$ and $\mathbf{W} \in \mathbb{R}^{n \times p}$. Here $\text{col}\{\cdot\}$ denotes the set of column vectors of the relevant linear operator. Without loss of generality, one can assume $\text{col}\{\mathbf{W}\} \subseteq \text{col}\{\mathbf{B}\}$ if $n \geq p$. By the inclusion $\text{col}\{\mathbf{W}\} \subseteq \text{col}\{\mathbf{B}\}$, the compactness of $\mathcal{C}_\tau$ and the unboundedness of $d\mathbf{\Gamma}_b$ the relation:

$$d\mathbf{\Gamma} = d\mathbf{\Gamma}_b + \boldsymbol{\tau} dt, \quad (2.126)$$

is tractable.

2.5 Complementarity Description of the Set-valued Impulsive Blocking Control Action

Blocking is investigated in two dimensions. The first dimension incorporates the action of blocking itself where a relative velocity is reduced immediately to zero. The second dimension

is the action of keeping blocked. The first dimension involves impulsive control that acts on the system at an instant and the second one a conventional control action on force level over an interval such that a relative velocity i.e. relative acceleration is kept zero. The impulsive and the conventional control action have set-valued character, in the sense that beyond a threshold that is necessary to block impactively or to keep blocked, any arbitrary value of the impulsive and conventional blocking control action is possible in the complementarity framework. The impulsive set-valued blocking control law is described by the two set-valued relations in (2.121) and (2.122) which are depicted in figures 2.6 and 2.9. The differential measure of the normal control force can be seen as composed of a Lebesgue measurable and a Borel measurable part. In this case, relation (2.122) is visualised as in the figures 2.8 and 2.9. The differential measures of the normal control force dN , right normal control force dN_r and left normal control force dN_l are decomposed into Lebesgue and Borel measurable parts as given in equations (2.127) to (2.129):

$$dN = n dt + N' d\sigma = n dt + (N^+ - N^-) d\sigma, \quad (2.127)$$

$$dN_r = n_r dt + N'_r d\sigma = n_r dt + (N_r^+ - N_r^-) d\sigma, \quad (2.128)$$

$$dN_l = n_l dt + N'_l d\sigma = n_l dt + (N_l^+ - N_l^-) d\sigma. \quad (2.129)$$

Here N' , N'_r and N'_l denote the Radon-Nykodym derivatives of dN , dN_r and dN_l , respectively. The differential measures of the blocking control force $d\Gamma$, right normal control force $d\Gamma_r$ and left normal control force $d\Gamma_l$ are decomposed into Lebesgue and Borel measurable parts as given in equations (2.130) to (2.132):

$$d\Gamma_b = \tau_b dt + \Gamma'_b d\sigma = \tau_b dt + (\Gamma_b^+ - \Gamma_b^-) d\sigma, \quad (2.130)$$

$$d\Gamma_{br} = \tau_{br} dt + \Gamma'_{br} d\sigma = \tau_{br} dt + (\Gamma_{br}^+ - \Gamma_{br}^-) d\sigma, \quad (2.131)$$

$$d\Gamma_{bl} = \tau_{bl} dt + \Gamma'_{bl} d\sigma = \tau_{bl} dt + (\Gamma_{bl}^+ - \Gamma_{bl}^-) d\sigma. \quad (2.132)$$

Here Γ' , Γ'_{br} and Γ'_{bl} denote the Radon-Nykodym derivatives of $d\Gamma$, $d\Gamma_{br}$ and $d\Gamma_{bl}$, respectively. The evaluation of the Lebesgue-Stieltjes integral of dN and $d\Gamma$ over an atomic instant of time where an impulsive control action is applied on the system yields:

$$\int_{\{t_i\}} dN = N^+ - N^- = \hat{N}, \quad (2.133)$$

$$\int_{\{t_i\}} d\Gamma_b = \Gamma_b^+ - \Gamma_b^- = \hat{\Gamma}_b, \quad (2.134)$$

respectively. The entities $\hat{N}$ and $\hat{\Gamma}_b$ are denoted by impulsive normal blocking force and impulsive control force. The decomposition of the UPR (unilateral primitive)-type relation between $|\gamma_b^+|$ and $\hat{N}$, into two UPR relations as depicted in figure 2.6 is expressed by set of relations

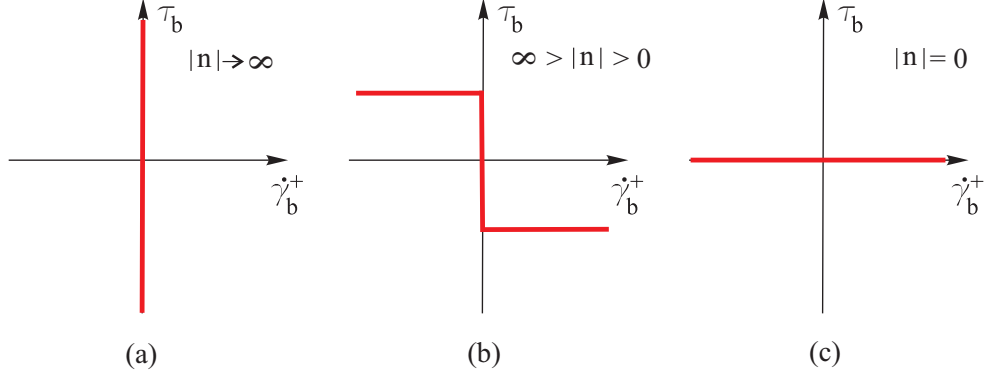


Figure 2.5: (a) A perfect bilateral constraint is equivalent to a normal control force that takes infinite values, (b) For values of N less than infinity and greater zero, the blocking action has friction characteristics, (c) if $n = 0$ there is no braking force.

(2.135) to (2.138):

$$\hat{N} = \hat{N}_r + \hat{N}_l, \quad (2.135)$$

$$\gamma_b^+ = \gamma_{br}^+ - \gamma_{bl}^+, \quad (2.136)$$

$$\gamma_{br}^+ \hat{N}_r = 0, \quad \hat{\Gamma}_{br}^+ \geq 0, \quad \hat{N}_r \geq 0, \quad (2.137)$$

$$\gamma_{bl}^+ \hat{N}_l = 0, \quad \hat{\Gamma}_{bl}^+ \geq 0, \quad \hat{N}_l \geq 0. \quad (2.138)$$

In a nonimpactive phase of motion, the action of keeping blocked or applying no blocking force is formulated as an UPR between $|\dot{\gamma}_b^+|$ and N . The decomposition of the UPR-type relation between $|\dot{\gamma}_b^+|$ and N , into two UPR relations as depicted in figure 2.11 is expressed by set of relations (2.139) to (2.142):

$$n = n_r + n_l, \quad (2.139)$$

$$\dot{\gamma}_b^+ = \dot{\gamma}_{br}^+ - \dot{\gamma}_{bl}^+, \quad (2.140)$$

$$\dot{\gamma}_{br}^+ n_r = 0, \quad \dot{\gamma}_{br}^+ \geq 0, \quad n_r \geq 0, \quad (2.141)$$

$$\dot{\gamma}_{bl}^+ n_l = 0, \quad \dot{\gamma}_{bl}^+ \geq 0, \quad n_l \geq 0. \quad (2.142)$$

The relations given in (2.135) to (2.138) are equivalently expressed as:

$$\gamma_b^+ = \gamma_{br}^+ - \gamma_{bl}^+, \quad (2.143)$$

$$\gamma_{bl}^+ \hat{N} = 0, \quad \hat{\Gamma}_{bl}^+ \geq 0, \quad \hat{N} \geq 0, \quad (2.144)$$

$$\gamma_{br}^+ \hat{N} = 0, \quad \hat{\Gamma}_{br}^+ \geq 0, \quad \hat{N} \geq 0. \quad (2.145)$$

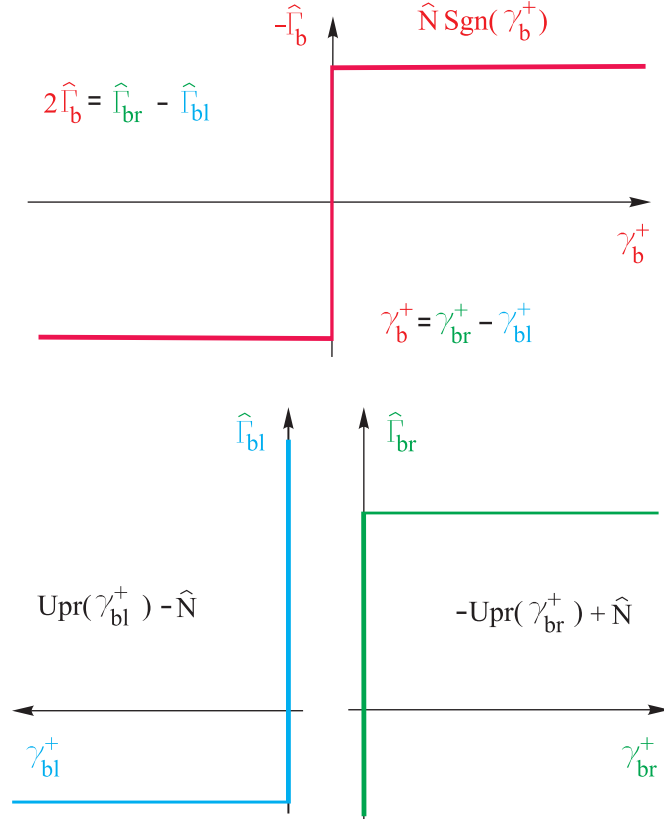


Figure 2.6: The set-valued signum relation and its decomposition into two unilateral primitives that represents relation between the discretised differential measure of normal blocking force to the discretised differential measure of the control force.

The relations given in (2.139) to (2.142) are equivalently expressed as:

$$\dot{\gamma}_b^+ = \dot{\gamma}_{br}^+ - \dot{\gamma}_{bl}^+, \quad (2.146)$$

$$\dot{\gamma}_{bl}^+ n = 0, \dot{\gamma}_{bl}^+ \geq 0, \quad n \geq 0, \quad (2.147)$$

$$\dot{\gamma}_{br}^+ n = 0, \dot{\gamma}_{br}^+ \geq 0, \quad n \geq 0. \quad (2.148)$$

The figure 2.6 shows the relation between the differential measures of normal control force and the blocking control force. It depicts the decomposition of the set-valued signum characteristics into two unilateral primitives graphically, which are stated in the relations (2.149) to

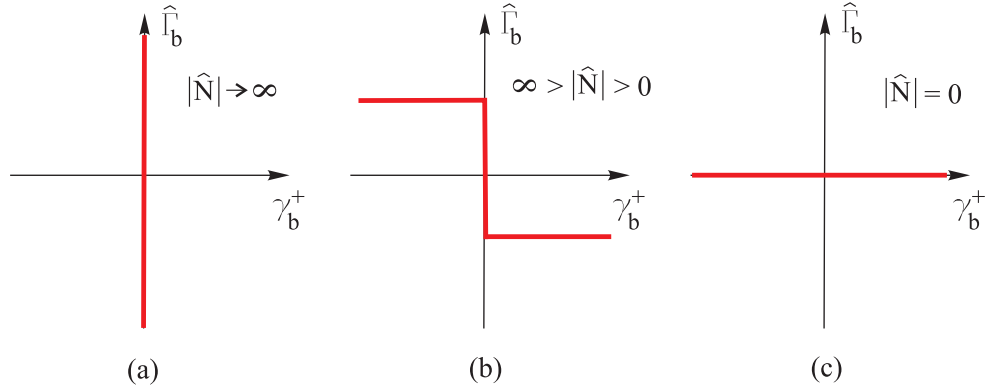


Figure 2.7: (a) If $\hat{N}$ rises to infinity, the relative velocity in the next moment drops to zero, (b) For values of $\hat{N}$ less than infinity and greater zero, the blocking action has friction characteristics with respect to the relative velocity, (c) if $\hat{N} = 0$ there is no blocking in the next moment.

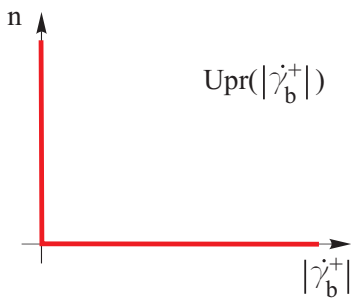


Figure 2.8: The unilateral primitive about the relation of the blocking normal force to the relative acceleration $\dot{\gamma}_b^+$.

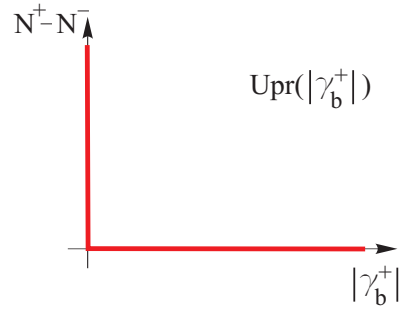


Figure 2.9: The UPR about the relation of the differential measure of blocking normal impulsive force to the relative velocity $\gamma_b^+ \in \mathcal{RCLBV}$.

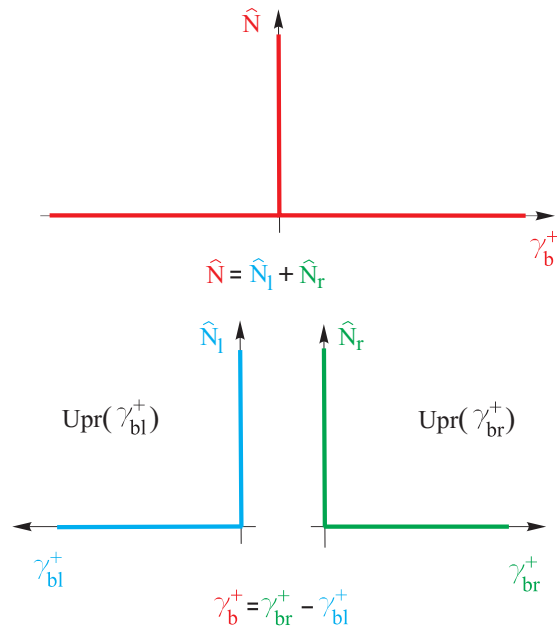


Figure 2.10: The decomposition of the Upr relation in figure 2.9 into two unilateral primitives.

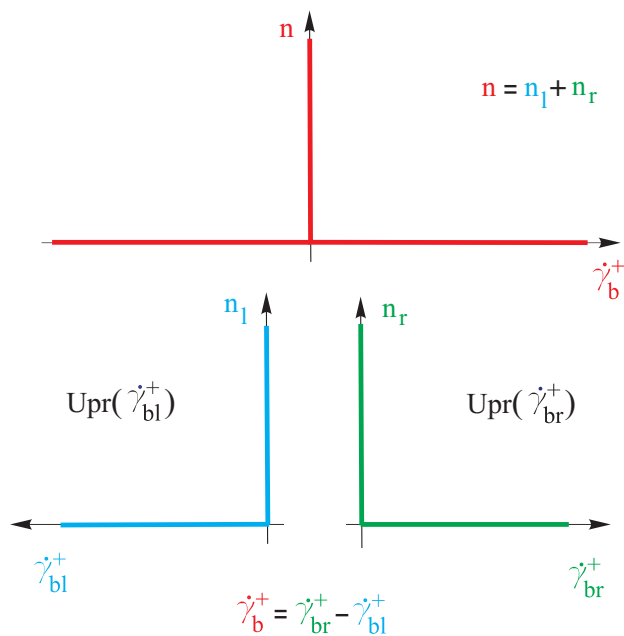


Figure 2.11: The decomposition of the Upr relation in figure 2.8 into two unilateral primitives.

(2.153):

$$\hat{\Gamma}_{\text{br}_i} = \hat{\Gamma}_{\text{b}_i} + \hat{N}_i, \quad \forall i \in \mathcal{I}_B, \quad (2.149)$$

$$\hat{\Gamma}_{\text{bl}_i} = -\hat{\Gamma}_{\text{b}_i} + \hat{N}_i, \quad \forall i \in \mathcal{I}_B, \quad (2.150)$$

$$\gamma_{\text{b}_i}^+ = \gamma_{\text{br}_i}^+ - \gamma_{\text{bl}_i}^+ \quad \forall i \in \mathcal{I}_B, \quad (2.151)$$

$$\gamma_{\text{br}_i}^+ \hat{\Gamma}_{\text{br}_i} = 0, \quad \hat{\Gamma}_{\text{br}_i} \geq 0, \quad \gamma_{\text{br}_i}^+ \geq 0, \quad \forall i \in \mathcal{I}_B, \quad (2.152)$$

$$\gamma_{\text{bl}_i}^+ \hat{\Gamma}_{\text{bl}_i} = 0, \quad \hat{\Gamma}_{\text{bl}_i} \geq 0, \quad \gamma_{\text{bl}_i}^+ \geq 0 \quad \forall i \in \mathcal{I}_B. \quad (2.153)$$

In a nonimpactive phase of motion, the signum characteristics is expressed by the set of relations (2.154) to (2.158):

$$\tau_{\text{br}} = \tau_{\text{b}} + n, \quad (2.154)$$

$$\tau_{\text{bl}} = -\tau_{\text{b}} + n, \quad (2.155)$$

$$\dot{\gamma}_{\text{b}}^+ = \dot{\gamma}_{\text{br}}^+ - \dot{\gamma}_{\text{bl}}^+, \quad (2.156)$$

$$\dot{\gamma}_{\text{br}}^+ \tau_{\text{br}} = 0, \quad \dot{\gamma}_{\text{br}}^+ \geq 0, \quad \tau_{\text{br}} \geq 0, \quad (2.157)$$

$$\dot{\gamma}_{\text{bl}}^+ \tau_{\text{bl}} = 0, \quad \dot{\gamma}_{\text{bl}}^+ \geq 0, \quad \tau_{\text{bl}} \geq 0. \quad (2.158)$$

The relations given from (2.143) to (2.145) and from (2.149) to (2.153) are combined in (2.159) to (2.165) to reformulate the impulsive set-valued blocking control law given in equations (2.121) and (2.122) in the complementarity framework:

$$\hat{\Gamma}_{\text{br}_i} = \hat{\Gamma}_{\text{b}_i} + \hat{N}_i, \quad \forall i \in \mathcal{I}_B, \quad (2.159)$$

$$\hat{\Gamma}_{\text{bl}_i} = -\hat{\Gamma}_{\text{b}_i} + \hat{N}_i, \quad \forall i \in \mathcal{I}_B, \quad (2.160)$$

$$\gamma_{\text{b}_i}^+ = \gamma_{\text{br}_i}^+ - \gamma_{\text{bl}_i}^+ \quad \forall i \in \mathcal{I}_B, \quad (2.161)$$

$$\gamma_{\text{br}_i}^+ \hat{\Gamma}_{\text{br}_i} = 0, \quad \hat{\Gamma}_{\text{br}_i}^+ \geq 0, \quad \hat{\Gamma}_{\text{br}_i} \geq 0, \quad \forall i \in \mathcal{I}_B, \quad (2.162)$$

$$\gamma_{\text{bl}_i}^+ \hat{\Gamma}_{\text{bl}_i} = 0, \quad \hat{\Gamma}_{\text{bl}_i}^+ \geq 0, \quad \hat{\Gamma}_{\text{bl}_i} \geq 0, \quad \forall i \in \mathcal{I}_B, \quad (2.163)$$

$$\gamma_{\text{br}_i}^+ \hat{N}_i = 0, \quad \hat{\Gamma}_{\text{br}_i}^+ \geq 0, \quad \hat{N}_i \geq 0, \quad \forall i \in \mathcal{I}_B, \quad (2.164)$$

$$\gamma_{\text{bl}_i}^+ \hat{N}_i = 0, \quad \hat{\Gamma}_{\text{bl}_i}^+ \geq 0, \quad \hat{N}_i \geq 0, \quad \forall i \in \mathcal{I}_B. \quad (2.165)$$

In a similar way, the behaviour of the set-valued controls in the absence of impulsive action is captured by the combination of the relations given from (2.146) to (2.148) and from (2.154) to (2.158) are combined in (2.166) to (2.172) in order to reformulate the set-valued blocking

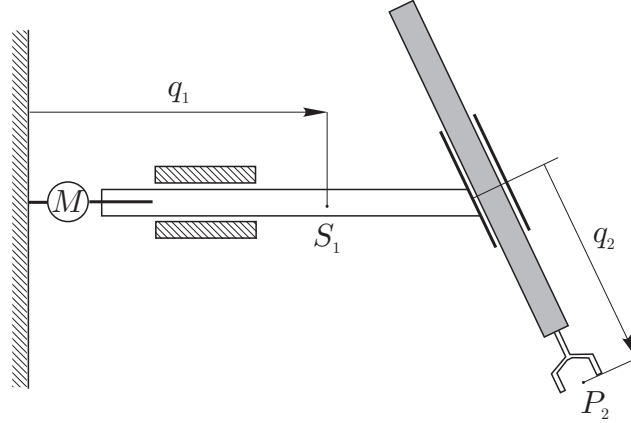


Figure 2.12: The 2-DOF linear mechanism

control in the complementarity framework on acceleration level:

$$\tau_{br_i} = \tau_{b_i} + n_i, \quad \forall i \in \mathcal{I}_B, \quad (2.166)$$

$$\tau_{bl_i} = -\tau_{b_i} + n_i, \quad \forall i \in \mathcal{I}_B, \quad (2.167)$$

$$\dot{\gamma}_{b_i}^+ = \dot{\gamma}_{br_i}^+ - \dot{\gamma}_{bl_i}^+, \quad \forall i \in \mathcal{I}_B, \quad (2.168)$$

$$\dot{\gamma}_{br_i}^+ \tau_{br_i} = 0, \quad \dot{\gamma}_{br_i}^+ \geq 0, \quad \tau_{br_i} \geq 0, \quad \forall i \in \mathcal{I}_B, \quad (2.169)$$

$$\dot{\gamma}_{bl_i}^+ \tau_{bl_i} = 0, \quad \dot{\gamma}_{bl_i}^+ \geq 0, \quad \tau_{bl_i} \geq 0, \quad \forall i \in \mathcal{I}_B, \quad (2.170)$$

$$\dot{\gamma}_{br_i}^+ n_i = 0, \quad \dot{\gamma}_{br_i}^+ \geq 0, \quad n_i \geq 0, \quad \forall i \in \mathcal{I}_B, \quad (2.171)$$

$$\dot{\gamma}_{bl_i}^+ n_i = 0, \quad \dot{\gamma}_{bl_i}^+ \geq 0, \quad n_i \geq 0, \quad \forall i \in \mathcal{I}_B. \quad (2.172)$$

This modeling approach of set-valued impulsive blocking control is published in [115] by Yunt *et al.*.

2.5.1 A Case Study

At this point it is appropriate to illustrate this characteristic by making use of an example mechanical system. The example mechanical system is a linear planar mechanical system with two-degrees of freedom (DOF) as depicted in figure 2.12. One DOF of the system is fully actuated and the inclined link can only be commanded by frictional braking. This example is considered as the most simple nonsmooth underactuated mechanical system with explicit phase transitions. The time-optimal control problem of this measure-differential inclusion is stated

as follows:

$$\min \quad t_f \quad (2.173)$$

subject to boundary constraints

$$q_1(t_f) = q_{1f}, \quad q_2(t_f) = q_{2f}, \quad u_1(t_f) = u_{1f}, \quad u_2(t_f) = u_{2f},$$

Set-Valued Mixed Lebesgue-Borel type Control Constraints

$$d\Lambda_s \in -d\Gamma \operatorname{Sgn}(\xi_s),$$

Constraints on the ordinary control

$$\tau_{\min} \leq \tau \leq \tau_{\max} \quad \forall t, \quad (2.174)$$

and to the dynamics in measure-differential equation form:

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos(\alpha) \\ m_2 \cos(\alpha) & m_2 \end{bmatrix} \begin{bmatrix} du_1 \\ du_2 \end{bmatrix} = \begin{bmatrix} \tau dt \\ d\Lambda_s \end{bmatrix}. \quad (2.175)$$

where ξ_s is given by $\xi_s = u_2^+ + \epsilon_s u_2^-$. Without loss of generality the tangential restitution coefficient is taken zero. The friction law is decomposed into two upper primitives resulting in two sets of complementary relations. These relations are given by:

$$u_2^+ = \xi_r - \xi_l, \quad (2.176)$$

$$d\Lambda_s = d\Lambda_r - d\Lambda_l, \quad (2.177)$$

$$d\Gamma = d\Lambda_r + d\Lambda_l, \quad (2.178)$$

$$d\Lambda_r \geq 0, \quad d\Lambda_l \geq 0, \quad (2.179)$$

$$\xi_r \geq 0, \quad \xi_l \geq 0, \quad (2.180)$$

$$\xi_r d\Lambda_r = 0, \quad \xi_l d\Lambda_l = 0, \quad (2.181)$$

$$\xi_r d\Gamma = 0, \quad \xi_l d\Gamma = 0. \quad (2.182)$$

By making use equations (2.176), (2.177), (2.178) several variables in the complementarities in (2.181) and (2.182) are eliminated.

2.6 Passively Actuated Robotic Manipulators with blockable DOF

The difference in the structure of the equations of motion in different phases of motion arises from dimension of admissible variations. The representation of rigid-body mechanical dynamical systems can be adjusted in three ways to a change in the dimension of admissible variations as summarised below:

- The number of algebraical equations in the differential-algebraical representation can be changed. In this case there exists a set of algebraic equations that have to be fulfilled during the integration of the differential equations.
- A new set of generalised coordinates can be selected, to eliminate the necessity of taking care of the algebraical constraints.
- The equations of motion can be projected to a subspace such that the dynamics do not evolve in the restrained directions of motion, without changing the number of generalised coordinates or generating additional algebraical constraints.

The modeling approach presented in this section is published in short form in [111] by Yunt.

2.6.1 Post-transition Equations of Motion and Discontinuity Conditions

The assumptions during a possibly impactive transition are given as follows:

- The transitions which involve blocking a relative joint DOF may be impactively, which arises from fully inelastic impacts.
- The generalised position remains unchanged during transition.
- The impulsive control action, when triggered reduces the post-impact relative joint velocity to zero.

Further, the height of the set-valued signum relation must be high enough to reduce the relative velocity to zero immediately and must be zero if unblocked. Let n is the total number of degrees of freedom of the robotic manipulator. The equations of motion are given by:

$$\mathbf{M}(\mathbf{q}) \dot{\mathbf{u}} - \mathbf{h}(\mathbf{q}, \mathbf{u}) - \mathbf{B}(\mathbf{q}) \boldsymbol{\tau} = \mathbf{0}, \quad (2.183)$$

where $\mathbf{q}$ and $\mathbf{u}$ denote the absolutely continuous generalised positions and bounded variation generalised velocities, respectively. Here $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n \times n}$ is the mass matrix, $\mathbf{h}(\mathbf{q}, \mathbf{u}) \in \mathbb{R}^{n \times 1}$

denote the vector of gyroscopical and coriolis, smooth potential (gravity, spring etc.) forces and $\boldsymbol{\tau} \in \mathbb{R}^{s \times 1}$ are the controls which can be set-valued. The linear operator $\mathbf{B}(\mathbf{q}) \in \mathbb{R}^{n \times s}$ includes the generalised control directions. This representation is the maximal representation, meaning that all joints are unblocked. The relative joint velocity at any joint is given by (2.184):

$$\boldsymbol{\gamma}_i = \mathbf{w}_i^T(\mathbf{q}) \mathbf{u}. \quad (2.184)$$

For such a Lagrangian system a n dimensional vector $\boldsymbol{\gamma} \in \mathbb{R}^n$ is defined:

$$\boldsymbol{\gamma} = \mathbf{W}^T(\mathbf{q}) \mathbf{u}, \quad (2.185)$$

such that $\mathbf{w}_i(\mathbf{q}) \in \text{col}\{\mathbf{W}\}$ and $\mathbf{W} \in \mathbb{R}^{n \times n}$. Here $\text{col}\{\cdot\}$ denotes the set of column vectors of the relevant linear operator. In order to describe the transition condition at any $t_i \in \mathcal{I}_T$ properly following index sets are defined:

$$\begin{aligned} \mathcal{C}_{B_i} &= \{i \mid \gamma_i^+ = 0, \dot{\gamma}_i^+ = 0, \forall t \in (t_i^+, t_{i+1}^-)\}, \\ \mathcal{C}_{P_i} &= \{i \mid \gamma_i^+ = 0, \dot{\gamma}_i^+ \in \mathbb{R}\}. \end{aligned}$$

Here $\mathcal{C}_{B_i}$ denotes the index set of joints that remain blocked after a transition time until another possible transition time. The set $\mathcal{C}_{P_i}$ is the index set of joints at which a blocking action takes place at a transition time, such that the relative post-transition joint velocity is nullified. Let $\mathcal{C}_{P_i \setminus B_i}$ be defined as:

$$\mathcal{C}_{P_i \setminus B_i} = \{i \mid i \in \mathcal{C}_{P_i}, i \notin \mathcal{C}_{B_i}\}. \quad (2.186)$$

For convenience it is assumed that $\mathcal{C}_{P_i \setminus B_i} = \emptyset$ without loss of generality. Let $p < s$ the number DOF which are being blocked impactively. The impact equation is given by the following expression:

$$\mathbf{M}(\mathbf{q}) (\mathbf{u}^+ - \mathbf{u}^-) - \mathbf{W}_b(\mathbf{q}) \boldsymbol{\Gamma} = \mathbf{0}, \quad (2.187)$$

where $\boldsymbol{\Gamma} \in \mathbb{R}^p$ are blocking impulses that can be generated at the joints, which participate in blocking and the matrix $\mathbf{W}_b \in \mathbb{R}^{n \times p}$ denotes the generalised force direction of the blocking forces, such that $\text{col}\{\mathbf{W}_b\} \subset \text{col}\{\mathbf{B}\}$. Further, it is assumed that $\text{col}\{\mathbf{B}\} \subset \text{col}\{\mathbf{W}\}$ for convenience and without loss of generality. The difference between the pre-impact and post-impact relative joint contact velocities is related to the post-, and pre-impact generalised velocities of the mechanical system by expression (2.188):

$$\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^- = \mathbf{W}^T(\mathbf{q}) (\mathbf{u}^+ - \mathbf{u}^-). \quad (2.188)$$

Let at a transition, which is accompanied by an impact, which is induced by the sudden blocking of directions of motion, p of the joints, characterised by their force directions, be active. Then, the vector $\boldsymbol{\gamma}$ can be decomposed in the following manner:

$$\begin{bmatrix} \gamma_b^+ \\ \vdots \\ \gamma_f^+ \end{bmatrix} - \begin{bmatrix} \gamma_b^- \\ \vdots \\ \gamma_f^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_b & \vdots & \mathbf{W}_f \end{bmatrix}^T (\mathbf{u}^+ - \mathbf{u}^-), \quad (2.189)$$

where γ_b^+ and γ_b^- denote the relative joint post-, and pretransition velocities at the blocked/active joints, and γ_f^+ and γ_f^- denote the relative joint post-, and pretransition velocities at the free/passive joints. Here $\mathbf{W}_b \in \mathbb{R}^{n \times p}$, $\mathbf{W}_f \in \mathbb{R}^{n \times (n-p)}$ denote the matrices, consisting column-wise of blocked and unblocked generalised directions such that $\text{col}\{\mathbf{W}_f\} \cup \text{col}\{\mathbf{W}_b\} = \text{col}\{\mathbf{W}\}$ and $\text{col}\{\mathbf{W}_f\} \cap \text{col}\{\mathbf{W}_b\} = \emptyset$. The equation (2.187) can be solved for the jump in the generalised velocities of the system:

$$\mathbf{u}^+ - \mathbf{u}^- = \mathbf{M}^{-1}(\mathbf{q}) \mathbf{W}_b(\mathbf{q}) \boldsymbol{\Gamma}. \quad (2.190)$$

Inserting this expression in (2.188) reveals the jump in the vector of relative joint velocity vector:

$$\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^- = \mathbf{W}^T(\mathbf{q}) (\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{W}^T \mathbf{M}^{-1} \mathbf{W} \begin{bmatrix} \boldsymbol{\Gamma} \\ \dots \\ \boldsymbol{\Delta} \end{bmatrix}. \quad (2.191)$$

By making use of the decomposition of the relative joint velocities into blocked and free directions as introduced in equation (2.189) following is obtained:

$$\begin{bmatrix} \gamma_b^+ - \gamma_b^- \\ \dots \\ \gamma_f^+ - \gamma_f^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{W}_b & \vdots & \mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{W}_f \\ \dots & \dots & \dots \\ \mathbf{W}_f^T \mathbf{M}^{-1} \mathbf{W}_b & \vdots & \mathbf{W}_f^T \mathbf{M}^{-1} \mathbf{W}_f \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma} \\ \dots \\ \boldsymbol{\Delta} \end{bmatrix}. \quad (2.192)$$

In order to simplify the notation matrices $\mathbf{G}_{bb}$, $\mathbf{G}_{bf}$, $\mathbf{G}_{fb}$ and $\mathbf{G}_{ff}$ are introduced as follows:

$$\begin{bmatrix} \gamma_b^+ - \gamma_b^- \\ \dots \\ \gamma_f^+ - \gamma_f^- \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{bb} & \vdots & \mathbf{G}_{bf} \\ \dots & \dots & \dots \\ \mathbf{G}_{fb} & \vdots & \mathbf{G}_{ff} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma} \\ \dots \\ \boldsymbol{\Delta} \end{bmatrix}. \quad (2.193)$$

The blocking at every direction is induced by a single control direction. Further, immediately after blocking action the post-impact relative velocity γ^+ immediately reduces to zero. Given these two assumptions the impulse at contacts which do not participate at the blocking and post-transition velocity at the blocked joints are zero, as stated below:

$$\boldsymbol{\Delta} = \mathbf{0}, \quad (2.194)$$

$$\gamma_b^+ = \mathbf{0}. \quad (2.195)$$

The impulse vector $\boldsymbol{\Gamma}$ can be eliminated after insertion of (2.194) and (2.195) into (2.193), which boils down to:

$$-\gamma_b^- = \mathbf{G}_{bb} \boldsymbol{\Gamma}, \quad (2.196)$$

$$\gamma_f^+ - \gamma_f^- = \mathbf{G}_{fb} \boldsymbol{\Gamma}. \quad (2.197)$$

Equation (2.196) when solved for $\mathbf{\Gamma}$ reveals:

$$\mathbf{\Gamma} = -\mathbf{G}_{bb}^{-1} \boldsymbol{\gamma}_b^-, \quad (2.198)$$

and insertion into equation (2.197) eliminates the impulse and establishes the relation between post-, and pre-impact relative joint velocities:

$$\boldsymbol{\gamma}_f^+ = \boldsymbol{\gamma}_f^- - \mathbf{G}_{fb} \mathbf{G}_{bb}^{-1} \boldsymbol{\gamma}_b^-. \quad (2.199)$$

This equation can be rewritten in terms of the post-, and pre-impact generalised velocities by making use of equation (2.189) as given in (2.200):

$$\mathbf{W}_f^T \mathbf{u}^+ = \mathbf{W}_f^T \mathbf{u}^- - \mathbf{G}_{fb} \mathbf{G}_{bb}^{-1} \mathbf{W}_b^T \mathbf{u}^-, \quad (2.200)$$

and by defining $\mathbf{K}(\mathbf{q}) = \mathbf{W}_f^T - \mathbf{G}_{fb} \mathbf{G}_{bb}^{-1} \mathbf{W}_b^T$ reveals following expression:

$$\mathbf{W}_f^T(\mathbf{q}) \mathbf{u}^+ - \mathbf{K}(\mathbf{q}) \mathbf{u}^- = \mathbf{0}. \quad (2.201)$$

On the other hand by insertion of the impulse obtained in (2.198) into (2.190) reveals the relation between post- and pretransition generalised velocities:

$$\mathbf{u}^+ - \mathbf{u}^- = -\mathbf{M}^{-1} \mathbf{W}_b \mathbf{G}_{bb}^{-1} \mathbf{W}_b^T \mathbf{u}^-. \quad (2.202)$$

This equation can be rewritten in the following form:

$$\mathbf{u}^+ = (\mathbf{I} - \mathbf{M}^{-1} \mathbf{W}_b \mathbf{G}_{bb}^{-1} \mathbf{W}_b^T) \mathbf{u}^- = \mathbf{P}_\perp^T(\mathbf{q}) \mathbf{u}^-. \quad (2.203)$$

The value of impulse established in (2.198) represents the minimal value to induce full blocking at joint $i \in \mathcal{C}_{B_i}$.

2.6.2 Lagrangian Dynamics in Different Phases of Motion

After the possibly impactive transition the equations of motion on acceleration level may differ from the pre-transition equations of motion based on the closed directions of motion. It is assumed that the interaction of the Lagrangian system with the surroundings (unilateral contacts, etc.) do not interfere during the course of control action. The generalised acceleration of the finite-dimensional Lagrangian system when some DOF are closed by $\boldsymbol{\tau}_b$, is given by (2.204) on acceleration level:

$$\dot{\mathbf{u}} = \mathbf{M}^{-1}(\mathbf{q}) \mathbf{h}(\mathbf{q}, \mathbf{u}) + \mathbf{M}^{-1}(\mathbf{q}) \mathbf{W}_b(\mathbf{q}) \boldsymbol{\tau}_b + \mathbf{M}^{-1}(\mathbf{q}) \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}. \quad (2.204)$$

The controls $\boldsymbol{\tau}_b$ represent the forces which are required to constrain the vector field from evolving in the direction $\boldsymbol{\gamma}_b$. The linear operator $\mathbf{W}_b$ denotes the generalised force direction

of the constraining forces, such that $\text{col}\{\mathbf{W}_b\} \subset \text{col}\{\mathbf{B}\}$. The acceleration $\dot{\gamma}_b$ in the blocked directions must vanish, which means that:

$$\dot{\gamma}_b = \mathbf{W}_b^T \dot{\mathbf{u}} + \dot{\mathbf{W}}_b^T \mathbf{u} = \mathbf{0} . \quad (2.205)$$

The insertion of equation (2.204) in equation (2.205) reveals:

$$\mathbf{W}_b^T \dot{\mathbf{u}} + \dot{\mathbf{W}}_b^T \mathbf{u} = \mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{h} + \mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{W}_b \boldsymbol{\tau}_b + \mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{B} \boldsymbol{\tau} + \dot{\mathbf{W}}_b^T \mathbf{u} = \mathbf{0} . \quad (2.206)$$

The equation (2.206) can be solved for the blocking forces/moments as below:

$$\boldsymbol{\tau}_b = -(\mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{W}_b)^{-1} \left(\mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{h} + \mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{B} \boldsymbol{\tau} + \dot{\mathbf{W}}_b^T \mathbf{u} \right) . \quad (2.207)$$

Defining the projector $\mathbf{P}_\parallel$ as

$$\mathbf{P}_\parallel = \mathbf{W}_b (\mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{W}_b)^{-1} \mathbf{W}_b^T \mathbf{M}^{-1} \quad (2.208)$$

and inserting into equation (2.204) reveals the projected dynamics:

$$\mathbf{M} \dot{\mathbf{u}} - \mathbf{h} - \mathbf{P}_\parallel (\mathbf{h} + \mathbf{B} \boldsymbol{\tau}) + \mathbf{W}_b (\mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{W}_b)^{-1} \dot{\mathbf{W}}_b^T \mathbf{u} - \mathbf{B} \boldsymbol{\tau} = \mathbf{0} . \quad (2.209)$$

The equations of motion after the directions $\mathbf{W}_b$ are closed in the generalised coordinates can be rearranged as below:

$$\mathbf{M} \dot{\mathbf{u}} - \mathbf{P}_\perp \mathbf{h} - \mathbf{P}_\perp \mathbf{B} \boldsymbol{\tau} + \mathbf{W}_b (\mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{W}_b)^{-1} \dot{\mathbf{W}}_b^T \mathbf{u} = \mathbf{0} . \quad (2.210)$$

The new vector of coriolis and gyroscopical forces as well as the linear operator of generalised control directions can be redefined as:

$$\mathbf{h}_b = \mathbf{P}_\perp \mathbf{h} - \mathbf{W}_b (\mathbf{W}_b^T \mathbf{M}^{-1} \mathbf{W}_b)^{-1} \dot{\mathbf{W}}_b^T \mathbf{u} , \quad (2.211)$$

$$\mathbf{B}_b = \mathbf{P}_\perp \mathbf{B} , \quad (2.212)$$

where the orthogonal projector $\mathbf{P}_\perp$ to $\mathbf{P}_\parallel$ is defined as

$$\mathbf{P}_\perp = \mathbf{I} - \mathbf{P}_\parallel , \quad (2.213)$$

with $\mathbf{I}$ being an identity matrix of appropriate size. So the Lebesgue measurable projected dynamics in n degrees of freedom can be stated as:

$$\mathbf{M}(\mathbf{q}) \dot{\mathbf{u}} - \mathbf{h}_b(\mathbf{q}, \mathbf{u}) - \mathbf{B}_b(\mathbf{q}) \boldsymbol{\tau} = \mathbf{0} . \quad (2.214)$$

2.6.3 Change in Mechanical Energy and Impuls and Dissipation

The Lagrangian of FDLS is defined as:

$$L(\mathbf{q}, \mathbf{u}) = T(\mathbf{q}, \mathbf{u}) - V(\mathbf{q}) \quad (2.215)$$

Here $T(\mathbf{q}, \mathbf{u})$ denotes the total kinetic, and $V(\mathbf{q})$ the total smooth potential energy of the system. The total energy of the system is:

$$H(\mathbf{q}, \mathbf{u}) = T(\mathbf{q}, \mathbf{u}) + V(\mathbf{q}). \quad (2.216)$$

The differential measure of the energy H is given by:

$$dH(\mathbf{q}, \mathbf{u}) = \frac{dH}{dt} dt + (T^+ - T^-) d\sigma \quad (2.217)$$

If the Lebesgue-Stieltjes Integral of the differential measure of the total energy H over an atomic time instant is evaluated then one obtains:

$$\int_{\{t_i\}} dH = T^+ - T^-. \quad (2.218)$$

The Borel measurable part of H is related to the jump in kinetic energy:

$$T^+ - T^- = \frac{1}{2} \langle \mathbf{u}^+, \mathbf{M}(\mathbf{q}) \mathbf{u}^+ \rangle - \frac{1}{2} \langle \mathbf{u}^-, \mathbf{M}(\mathbf{q}) \mathbf{u}^- \rangle, \quad (2.219)$$

and is nonzero if and only if there is an impulsive action that induces a jump in the generalized velocities. The energy dissipation during an impactive transition is given by:

$$\frac{1}{2} (\langle \mathbf{u}^+, \mathbf{M}(\mathbf{q}) \mathbf{u}^+ \rangle - \langle \mathbf{u}^-, \mathbf{M}(\mathbf{q}) \mathbf{u}^- \rangle) = \quad (2.220)$$

$$\begin{aligned} & \frac{1}{2} \langle \mathbf{P}_\perp^T \mathbf{u}^-, \mathbf{M}(\mathbf{q}) \mathbf{P}_\perp^T \mathbf{u}^- \rangle - \frac{1}{2} \langle \mathbf{u}^-, \mathbf{M}(\mathbf{q}) \mathbf{u}^- \rangle = \frac{1}{2} \langle (\mathbf{P}_\perp - \mathbf{I})^T \mathbf{u}^-, \mathbf{M}(\mathbf{q}) (\mathbf{P}_\perp^T + \mathbf{I}) \mathbf{u}^- \rangle \\ & = -\frac{1}{2} \langle \mathbf{W}_b^T \mathbf{u}^-, \mathbf{G}_{bb}^{-1} \mathbf{W}_b^T \mathbf{u}^- \rangle \end{aligned} \quad (2.221)$$

$$= -\frac{1}{2} \langle \gamma_b^-, \mathbf{G}_{bb}^{-1} \gamma_b^- \rangle. \quad (2.222)$$

Given this setting a straight-forward calculation shows that the change in total momentum $\mathbf{L}$ is given by:

$$\mathbf{L}^+ - \mathbf{L}^- = \mathbf{M}(\mathbf{q})(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{M}(\mathbf{q}) (\mathbf{P}_\perp^T - \mathbf{I}) \mathbf{u}^- = -\mathbf{M}(\mathbf{q}) \mathbf{P}_\parallel^T \mathbf{u}^- = -\mathbf{W}_b \mathbf{G}_{bb}^{-1} \mathbf{W}_b^T \mathbf{u}^-. \quad (2.223)$$

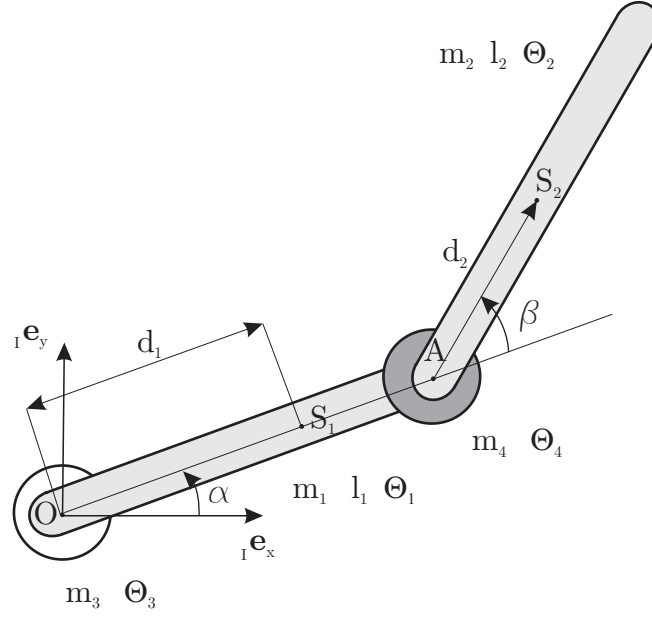


Figure 2.13: Planar double pendulum with one impactively blockable degrees of freedom.

2.6.4 Example: Planar Double Pendulum with One Blockable DOF

The example mechanical system can be seen in fig. 2.13. The system has two degrees of freedom, namely, α and β . The DOF α is measured with respect to the coordinate frame I . The DOF β denotes the relative angular position of the second link with respect to the first link. The first link is actuated by a motor at point O , is controlled continuously by single-valued, ordinary controller. The second link can be blocked by a brake at the axis, where both links join each other. This control action is set-valued, impulsive and discontinuous. The center of mass of each link is located at positions S_1 and S_2 . The symbols m , l and Θ denote the mass, length and inertia moments of the respective link, where following parameters are defined:

$$\begin{aligned}
 p_1 &= m_1 d_1^2 + m_2 l_1^2 + m_2 d_2^2 + \Theta_1 + \Theta_2, \\
 p_2 &= 2 m_2 l_1 d_2, \quad p_3 = m_2 d_2^2 + \Theta_2, \\
 p_4 &= m_2 l_1 d_2, \quad p_5 = m_2 d_2^2 + \Theta_2, \\
 p_6 &= (p_3 p_2 - p_4^2), \quad p_7 = -2 p_4 p_5, \\
 p_8 &= p_3 p_1 - p_5^2.
 \end{aligned}$$

where the positions of the center of masses of the links are given by:

$${}_I\mathbf{r}_{S_1} = \begin{bmatrix} d_1 \sin(\alpha) \\ d_1 \cos(\alpha) \\ 0 \end{bmatrix}, \quad {}_I\mathbf{r}_{S_2} = \begin{bmatrix} l_1 \sin(\alpha) + d_2 \sin(\alpha + \beta) \\ l_1 \cos(\alpha) + d_2 \cos(\alpha + \beta) \\ 0 \end{bmatrix}, \quad (2.224)$$

and the velocities of the center of masses are obtained by direct time differentiation:

$${}^I\mathbf{v}_{S_1} = \begin{bmatrix} d_1 \cos(\alpha) \dot{\alpha} \\ -d_1 \sin(\alpha) \dot{\alpha} \\ 0 \end{bmatrix} \quad (2.225)$$

$${}^I\mathbf{v}_{S_2} = \begin{bmatrix} (l_1 \cos(\alpha) + d_2 \cos(\alpha + \beta)) \dot{\alpha} + d_2 \cos(\alpha + \beta) \dot{\beta} \\ (-l_1 \sin(\alpha) - d_2 \sin(\alpha + \beta)) \dot{\alpha} - d_2 \sin(\alpha + \beta) \dot{\beta} \\ 0 \end{bmatrix}. \quad (2.226)$$

The angular velocities of the links are given by:

$${}^I\boldsymbol{\omega}_{S_1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}, \quad {}^I\boldsymbol{\omega}_{S_2} = \begin{bmatrix} 0 \\ 0 \\ \dot{\beta} + \dot{\alpha} \end{bmatrix}. \quad (2.227)$$

The total kinetic energy of the manipulator is:

$$T = \frac{1}{2} (p_1 + p_2 \cos(\beta)) \dot{\alpha}^2 + \frac{1}{2} p_3 \dot{\beta}^2 + \dot{\alpha} \dot{\beta} (\cos(\beta) p_4 + p_5). \quad (2.228)$$

The equations of motion are given by:

$$\begin{bmatrix} p_1 + p_2 \cos(\beta) & \cos(\beta) p_4 + p_5 \\ \cos(\beta) p_4 + p_5 & p_3 \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} - \begin{bmatrix} \sin(\beta) (p_2 \dot{\alpha} + \dot{\beta} p_4) \dot{\beta} \\ -\frac{1}{2} p_2 \sin(\beta) \dot{\alpha}^2 \end{bmatrix} - \begin{bmatrix} \tau \\ \lambda_s \end{bmatrix} = \mathbf{0}. \quad (2.229)$$

The generalised force directions are given by:

$$\mathbf{W}_b = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad \mathbf{W}_f = \begin{bmatrix} 1 & 0 \end{bmatrix}^T. \quad (2.230)$$

The terms for G_{bb} , G_{bf} , G_{fb} , G_{ff} are given by:

$$G_{bb} = \frac{p_1 + p_2 \cos(\beta)}{p_6 (\cos(\beta))^2 + p_7 \cos(\beta) + p_8}, \quad (2.231)$$

$$G_{bf} = \frac{-\cos(\beta) p_4 - p_5}{p_6 (\cos(\beta))^2 + p_7 \cos(\beta) + p_8} = G_{fb}, \quad (2.232)$$

$$G_{ff} = \frac{p_3}{p_6 (\cos(\beta))^2 + p_7 \cos(\beta) + p_8}. \quad (2.233)$$

The linear operator $\mathbf{K}$ is as follows:

$$\mathbf{K} = \begin{bmatrix} 1 & \frac{\cos(\beta) p_4 + p_5}{p_1 + p_2 \cos(\beta)} \end{bmatrix}, \quad (2.234)$$

Given this setting, the dissipated energy in the system in case of blocking the underactuated second link and the change in the total momentum amounts to:

$$\Delta T = -\frac{p_6 (\cos(\beta))^2 + p_7 \cos(\beta) + p_8}{2(p_1 + p_2 \cos(\beta))} \dot{\beta}(t_i^-)^2, \quad (2.235)$$

and

$$\Delta \mathbf{L} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix} G_{bb} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}(t_i^-) \\ \dot{\beta}(t_i^-) \end{bmatrix} = - \begin{bmatrix} 0 \\ \frac{p_1 + p_2 \cos(\beta)}{p_6 (\cos(\beta))^2 + p_7 \cos(\beta) + p_8} \end{bmatrix} \dot{\beta}(t_i^-) \quad (2.236)$$

respectively. The projector in (2.203) becomes in this case:

$$\mathbf{P}_{\perp}^T = \begin{bmatrix} 1 & (\cos(\beta) p_4 + p_5) / (p_1 + p_2 \cos(\beta)) \\ 0 & 0 \end{bmatrix}. \quad (2.237)$$

Chapter 3

Numerical Methods for the Trajectory Optimisation of Hybrid Finite-Dimensional Lagrangian Systems

The numerical treatment of optimal control of hybrid finite-dimensional Lagrangian systems (HFDLS) belongs to the applied mathematics branch of **Mathematical Programs with Equilibrium Constraints** (MPEC). This chapter commences with the definitions and classes of MPECs and the discussion of the features of the TOHLS from the perspective of MPEC. Both methods are based on augmented Lagrangian methods and therefore a section is devoted to different numerical methods in solving MPECs with emphasis placed on the augmented Lagrangian approach. Given the importance of complementarity problems in section 3.1 a general overview is provided. There is already a vast literature for dealing with the solution of NLP's that have variational inequality type constraints. The key to the solution of such problems are reformulations, which rely on proposing suitable classes of functions with the aid of which one can reformulate NCP's or LCP's as equality constraints. Because of the importance of reformulation functions in section 3.1 several such functions are discussed. The existence and uniqueness of the dual multipliers and the normality of the nonlinear programming problem is discussed in the sequel for the type of MPEC that arise from the trajectory optimisation of hybrid finite-dimensional structure-variant Lagrangian systems. Based on this analysis two numerical methods are introduced. To the best knowledge of the author the augmented Lagrangian method is first used in the framework of this thesis in the nonsmooth approach to MPEC.

In [73] a MPEC is defined as an optimisation problem in which the essential constraints are defined by parametric variational inequality or complementarity systems. One of the many

representations of a MPEC is stated in abstract form as follows:

$$\min_{\mathbf{x}, \mathbf{z}} \quad f(\mathbf{x}, \mathbf{z}), \quad (3.1)$$

$$\mathbf{z} \in \mathcal{S}(\mathbf{x}), \quad (3.2)$$

$$\mathbf{x} \in \mathcal{U}_{ad}, \mathbf{z} \in \mathcal{Z}. \quad (3.3)$$

The problem stated in (3.1) to (3.3) includes a subclass of so-called bilevel programs, where $\mathcal{S}$ assigns each $\mathbf{x} \in \mathcal{U}_{ad}$ the solution of a "lower-level" optimisation problem. In the case where the complementarity system arises from FDLS without Coulomb-type friction force interaction, a so-called subclass of MPEC, namely, bilevel programs apply. In [63] by Moreau it has been shown that the determination of the accelerations of a mechanical system subject to unilateral constraints without friction can be represented as a dual quadratic programming problem. Further, it is shown that the generalisation of the Gauss' variational principle is valid in the case of unilateral constraints without friction. Glocker showed that a quadratic programming problem is obtained if Tresca type friction, where the normal force is decoupled from the tangential force, exists. For the case of controlled nonsmooth Lagrangian systems the optimisation problem takes the form as described by relations (2.78) to (2.81), which is the dual of the extended principle of Gauss in control form. Here the bilevel programming structure requires to find the active contact forces for given control forces $\boldsymbol{\tau}$. The control forces at every instant represent the variables of the higher level optimisation problem and the contact forces represent the lower level optimisation variables. To find an optimal trajectory for such a mechanical problem means embedding a quadratic programming problem in every instant of time. The equations of motion along with the linear-complementarity conditions constitute necessary Karush-Kuhn-Tucker conditions of optimality for the quadratic programming problem that derives from the extended principle of Gauss in the absence of Coulomb-type of friction. If Coulomb-type of friction exists, then a QP can not be formulated any more because of the lack of the bisymmetrical structure of the $LCP(\mathbf{A}, \mathbf{b})$. So if Coulomb type friction exists at the contacts, then the optimal control problem is subject to variational inequalities and the mechanical quadratic programming problem does not exist any more, which is a problem to be treated in a broader class of problems, which require that at every instant of time the linear complementarity problem described by the relations (2.74) to (2.77) is solved. In the case of occurrence of impulsive control and impulsive active contact forces, one can classify the impulsive control forces as the primal variables of the MPEC, and the impulsive active contact forces as the secondary variables of the mathematical program. If impulsive control forces and impulsive contact forces are applied on the finite-dimensional Lagrangian system a dual problem in the impulsive forces and controls is formulated if some requirements are fulfilled. These are:

- The choice of an impact law, that has the character of a constitutive law must be made,

in order to relate pre-impact velocities to post-impact velocities.

- The collision and impact configurations have to be geometrically, kinematically and kinetically consistent.

This optimisation problem which can be seen as the impulsive dual of the extended Gauss Principle in control form has the same bilevel structure as the optimisation problem described above on acceleration level, namely, the "secondary variables" contact impulses are supposed to be determined in dependence of the impulsive control action. The unification of both problems is achieved by formulating the evolution of the mechanical dynamics subject to set-valued force interactions with the surroundings and the impulsive set-valued actions of the control strategy as a measure-differential inclusion. By analogy, the measure-differential inclusion, that describes the dynamics as a balance of measures, can be considered as the necessary conditions of a "lower-level" optimisation problem represented by the saddle-region restraining set $\mathcal{S}$. In this setting, the differential measures of generalised velocities and the generalised positions, the differential measures of the active contact forces are the secondary variables, whereas the differential measure of controls constitute the higher level variables, which affect the solutions of the subordinated optimisation problem. In all cases as mentioned above, the controls are considered as the variables of the "higher-level" optimisation problem whereas the contact forces and states are variables of the "lower-level" (quasi-) optimisation problem. If, however, in any of the mentioned forces entail Coulomb type forces then the determination of the differential measures of the contact forces as well as the differential measures of velocities can not be formulated as a QP problem with inequality constraints at every instant of time. By the presence of Coulomb type friction and tangential-impact laws the variational inequalities that describe the contact conditions do not resemble the Karush-Kuhn-Tucker conditions of a QP like problem any more. In this case, one is left with solving these variational inequalities which are mostly represented as nonlinear complementarity problems (NCP) or linear complementarity problems (LCP). Armed with the knowledge of the numerical techniques and the complementarity modeling to solve systems of equalities and inequalities involving variational inequalities, mathematical programs with equilibrium constraints (MPECs) that arise from TOHLS and given in the abstract setting in (3.1) to (3.3) is cast in the form below:

$$\min \quad f(\mathbf{x}, \mathbf{y}) \quad (3.4)$$

$$\text{subject to } (\mathbf{x}, \mathbf{y}) \in \mathcal{Z} \subset \mathbb{R}^{m+n}, \quad (3.5)$$

$$\text{such that } \mathbf{y} \text{ solves } \text{VI}(\mathbf{C}(\mathbf{x}), \mathbf{F}(\mathbf{x}, \cdot)), \quad (3.6)$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ are decision variables, $\mathcal{Z}$ is a closed set, $\mathbf{C}$ is a set-valued mapping, and $\mathbf{f} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}$ and $\mathbf{F} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ are given functions. Here the constraints state that the variable $\mathbf{y}$ must be a solution to a variational inequality that is parameterised by $\mathbf{x}$.

In the framework of this thesis **direct search methods** are minimisation methods based on function value comparison without making any use of first-order information like the Nelder-Mead simplex method. By **first-order information** all informations on subdifferentials in general and gradients in the special are meant which can be obtained numerically or analytically. **Gradient-based methods** are minimisation methods which make use of **first-order information** such as steepest descent or conjugate gradient methods. **Iterative methods** are methods that attempt to solve **first-order information** explicitly for the primal variables and update dual variables in an alternating manner like some classes of primal-dual methods like the augmented Lagrangian. By the nature of complementarity problems at solution irregularities may exist independent of whether the augmented Lagrangian pertaining to the problem is smooth or not.

3.1 Preliminaries

The theory of nonsmooth analysis is based on convex analysis. An open ball with center $\mathbf{x} \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $\mathbb{B}(\mathbf{x}; r)$, that is,

$$\mathbb{B}(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\| < r\}. \quad (3.7)$$

The set $\bar{\mathbb{B}}$ denotes the closure of $\mathbb{B}$.

The closed line-segment joining $\mathbf{x}$ and $\mathbf{y}$ is given by:

$$[\mathbf{x}, \mathbf{y}] = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{z} = \lambda \mathbf{x} + (1 - \lambda) \mathbf{y}, \text{ for } 0 \leq \lambda \leq 1\}, \quad (3.8)$$

and $(\mathbf{x}, \mathbf{y})$ represents the open line segment. A set $\mathcal{S} \subset \mathbb{R}^n$ is said to be convex if

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in \mathcal{S} \quad (3.9)$$

whenever $\mathbf{x}$ and $\mathbf{y}$ are in $\mathcal{S}$ and $\lambda \in [0, 1]$. The geometrical interpretation is that if the two ends of a line segment are in the set $\mathcal{S}$ then the entire connecting line segment also lies in the set $\mathcal{S}$. If $\mathcal{S}_i \subset \mathbb{R}^n$ are convex sets for $i = 1, \dots, m$ then their intersection $\bigcap_{i=1}^m \mathcal{S}_i$ is also convex. A linear combination $\sum_{i=1}^k \lambda_i \mathbf{x}_i$ is called a convex combination of elements $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$ if each $\lambda_i > 0$ and $\sum_{i=1}^k \lambda_i = 1$. The intersection of all the convex sets containing a given subset $S \subset \mathbb{R}^n$ is called the convex hull of set S and it is denoted by $\text{conv } \mathcal{S}$. For any $\mathcal{S} \subset \mathbb{R}^n$ the set $\text{conv } \mathcal{S}$ consists of all the convex combinations of the elements of $\mathcal{S}$, such that,

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^k \lambda_i \mathbf{x}_i, \sum_{i=1}^k \lambda_i = 1, \mathbf{x}_i \in \mathcal{S}, \lambda_i \geq 0\}. \quad (3.10)$$

The convex hull of set $\mathcal{S}$ is the smallest convex set containing $\mathcal{S}$, and $\mathcal{S}$ is convex if and only if $\mathcal{S} = \text{conv } \mathcal{S}$. The convex hull of a compact set is compact. The power set of a given set $\mathcal{S}$ is

the set of all subsets of S and is denoted by $\mathcal{P}(S)$. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y}) \quad (3.11)$$

whenever $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$ and $\lambda \in [0, 1]$. The f is said to be strictly convex if in equation (3.11) strict inequality is valid on the open line-segment $\lambda \in (0, 1)$.

Definition 3.1.1 A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz continuous at $\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$ with a positive constant L if there exists a positive number ϵ such that

$$|f(\mathbf{y}) - f(\mathbf{z})| \leq L \|\mathbf{y} - \mathbf{z}\|, \quad \forall \mathbf{y}, \mathbf{z} \in \mathbb{B}(\mathbf{x}; \epsilon). \quad (3.12)$$

Lemma 3.1.1 A convex function is locally Lipschitz continuous at $\forall \mathbf{x} \in \mathbb{R}^n$.

Definition 3.1.2 A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is positively homogeneous if

$$f(\lambda \mathbf{x}) = \lambda f(\mathbf{x}) \quad (3.13)$$

for all $\lambda \geq 0$ and subadditive if

$$f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y}) \quad (3.14)$$

for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. A function is said to be sublinear if it is both positively homogeneous and subadditive. A sublinear function is always convex.

Definition 3.1.3: Upper Semicontinuity A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be upper semicontinuous at $\mathbf{x} \in \mathbb{R}^n$ if for every sequence $\{\mathbf{x}_k\}$ converging to $\mathbf{x}$ the following holds

$$\limsup_{k \rightarrow \infty} f(\mathbf{x}_k) \leq f(\mathbf{x}). \quad (3.15)$$

Definition 3.1.4: Lower Semicontinuity A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be lower semicontinuous if

$$f(\mathbf{x}) \leq \liminf_{k \rightarrow \infty} f(\mathbf{x}_k). \quad (3.16)$$

Definition 3.1.5: Differentiability A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be differentiable at $\mathbf{x} \in \mathbb{R}^n$ if there exists a vector $\nabla f(\mathbf{x}) \in \mathbb{R}^n$ and a function $\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for all $\mathbf{d} \in \mathbb{R}^n$

$$f(\mathbf{x} + \mathbf{d}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T \mathbf{d} + \|\mathbf{d}\| \epsilon(\mathbf{d}). \quad (3.17)$$

and $\epsilon(\mathbf{d}) \rightarrow 0$ whenever $\|\mathbf{d}\| \rightarrow 0$.

Definition 3.1.6: Gradient The vector is called the gradient vector of the function f at $\mathbf{x}$ and has the following formula

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)^T, \quad (3.18)$$

where the components $\frac{\partial f}{\partial x_i}$, $i = 1, \dots, n$ are called the partial derivatives of f .

Definition 3.1.7: Continuous Differentiability If the function is differentiable and all the partial derivatives are continuous, then the function is said to be continuously differentiable or smooth or equivalently,

$$f \in C^1(\mathbb{R}^n). \quad (3.19)$$

Definition 3.1.8: Directional Derivative (Gâteaux Derivative) The directional derivative of f at $\mathbf{x} \in \mathbb{R}^n$ in the direction $\mathbf{d} \in \mathbb{R}^n$ is defined as the following limit:

$$f'(\mathbf{x}; \mathbf{d}) = \lim_{t \downarrow 0} \frac{f(\mathbf{x} + t \mathbf{d}) - f(\mathbf{x})}{t}. \quad (3.20)$$

The directional derivative is positively homogeneous and subadditive and hence sublinear as a function of $\mathbf{d}$. If a function f is differentiable at $\mathbf{x}$, then the directional derivative exists in every direction $\mathbf{d} \in \mathbb{R}^n$ and

$$f'(\mathbf{x}; \mathbf{d}) = \langle \nabla f(\mathbf{x}), \mathbf{d} \rangle. \quad (3.21)$$

If, in addition, f is convex, then for all $\mathbf{y} \in \mathbb{R}^n$

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), (\mathbf{y} - \mathbf{x}) \rangle. \quad (3.22)$$

Definition 3.1.9: Twice Differentiability A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice differentiable at $\mathbf{x} \in \mathbb{R}^n$ if there exists a vector $\nabla f(\mathbf{x}) \in \mathbb{R}^n$, a symmetric matrix $\nabla^2 f(\mathbf{x}) \in \mathbb{R}^{n \times n}$, and a function $\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for all $\mathbf{d} \in \mathbb{R}^n$

$$f(\mathbf{x} + \mathbf{d}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{d} \rangle + \frac{1}{2} \langle \mathbf{d}, \nabla^2 f(\mathbf{x}) \mathbf{d} \rangle + \|\mathbf{d}\|^2 \epsilon(\mathbf{d}), \quad (3.23)$$

where $\epsilon(\mathbf{d}) \rightarrow 0$. The linear operator is called the Hessian matrix of the function f at $\mathbf{x}$ and is defined to consist of second partial derivatives of f . If the function is twice differentiable and all the second partial derivatives are continuous, then the function is said to be twice continuously differentiable ($f \in C^2(\mathbb{R}^n)$).

Definition 3.1.10: Subdifferential the convex case The subdifferential of a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $\mathbf{x} \in \mathbb{R}^n$ is the set $\partial f(\mathbf{x})$ of vectors $\boldsymbol{\xi} \in \mathbb{R}^n$ such that

$$\partial f(\mathbf{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \boldsymbol{\xi}, \mathbf{y} - \mathbf{x} \rangle, \forall \mathbf{y} \in \mathbb{R}^n\}. \quad (3.24)$$

Definition 3.1.11: Directional Derivative of a convex function Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function. Then the directional derivative $f'(\mathbf{x}; \mathbf{d})$ exists in every direction $\mathbf{d} \in \mathbb{R}^n$ and it satisfies

$$f'(\mathbf{x}; \mathbf{d}) = \liminf_{t > 0} \frac{f(\mathbf{x} + t \mathbf{d}) - f(\mathbf{x})}{t}. \quad (3.25)$$

Lemma 3.1.2 Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function. Then for all $\mathbf{x} \in \mathbb{R}^n$

$$1. \ f'(\mathbf{x}; \mathbf{d}) = \max \{ \langle \boldsymbol{\xi}, \mathbf{d} \rangle \mid \boldsymbol{\xi} \in \partial f(\mathbf{x}), \forall \mathbf{d} \in \mathbb{R}^n \},$$

$$2. \partial f(\mathbf{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f'(\mathbf{x}; \mathbf{d}) \geq \langle \boldsymbol{\xi}, \mathbf{d} \rangle \forall \mathbf{d} \in \mathbb{R}^n\}.$$

Since for locally Lipschitz continuous functions there does not necessarily exist any classical directional derivative, we first define a generalised directional derivative. Using this generalised directional derivative the generalisation of the subdifferential to nonconvex locally Lipschitz continuous functions can be formulated.

Definition 3.1.12: The Generalised Directional Derivative Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function at $\mathbf{x} \in \mathbb{R}^n$. The generalised directional derivative of f at $\mathbf{x}$ in the direction $\mathbf{d} \in \mathbb{R}^n$ is defined by:

$$f^\circ(\mathbf{x}; \mathbf{d}) = \limsup_{\substack{\mathbf{y} \rightarrow \mathbf{x} \\ t \downarrow 0}} \frac{f(\mathbf{y} + t\mathbf{d}) - f(\mathbf{y})}{t} \quad (3.26)$$

The generalised directional derivative always exists for locally Lipschitz continuous functions and, as a function of $\mathbf{d}$, it is sublinear.

Definition 3.1.13: The Generalised Subdifferential Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function at $\mathbf{x} \in \mathbb{R}^n$. Then the subdifferential of f at $\mathbf{x}$ is the set $\partial f(\mathbf{x})$ of vectors $\boldsymbol{\xi} \in \mathbb{R}^n$ such that

$$\partial f(\mathbf{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f^\circ(\mathbf{x}; \mathbf{d}) \geq \langle \boldsymbol{\xi}, \mathbf{d} \rangle \quad \forall \mathbf{d} \in \mathbb{R}^n\}. \quad (3.27)$$

Each vector $\boldsymbol{\xi}$ is called a subgradient of f at $\mathbf{x}$.

The properties of the subdifferential are given as follows:

Lemma 3.1.3 Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function at $\mathbf{x} \in \mathbb{R}^n$ with a Lipschitz constant L . Then

1. $f^\circ(\mathbf{x}; \mathbf{d}) = \max\{\langle \boldsymbol{\xi}, \mathbf{d} \rangle \mid \boldsymbol{\xi} \in \partial f(\mathbf{x}), \forall \mathbf{d} \in \mathbb{R}^n\},$
2. $\partial f(\mathbf{x})$ is a nonempty, convex, and compact set such that $\partial f(\mathbf{x}) \subset \mathbb{B}(0; L),$
3. The mapping $\partial f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathcal{P}(\mathbb{R}^n)$ is upper semicontinuous.

The subdifferential for locally Lipschitz continuous functions is a generalisation of the subdifferential for convex functions.

Theorem 3.1.1 (Rademacher) Let $\mathcal{S} \subset \mathbb{R}^n$ be an open set. A function $f : \mathcal{S} \rightarrow \mathbb{R}$ that is locally Lipschitz continuous on $\mathcal{S}$ is differentiable almost everywhere on $\mathcal{S}$.

Proposition 3.1.1 Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz continuous at $\mathbf{x} \in \mathbb{R}^n$. Then

$$\partial f(\mathbf{x}) = \text{conv}\{\boldsymbol{\xi} \in \mathbb{R}^n \mid \text{there exists } \{\mathbf{x}_i\} \subset \mathbb{R}^n \setminus \Omega_f \text{ such that } \mathbf{x}_i \rightarrow \mathbf{x} \wedge \nabla f(\mathbf{x}_i) \rightarrow \boldsymbol{\xi}\}. \quad (3.28)$$

The set of points at which a given function f fails to be differentiable is denoted by Ω_f .

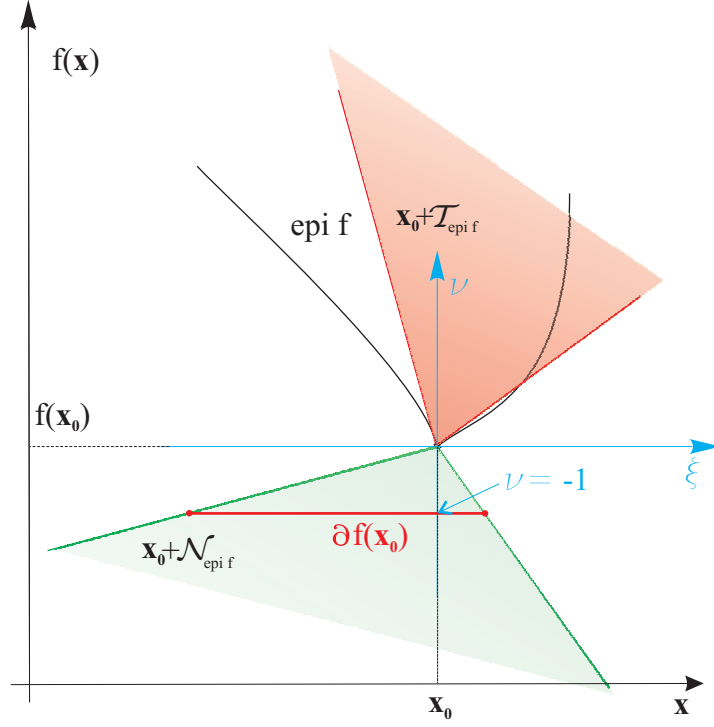


Figure 3.1: A nonconvex function f , its epigraph, and the Generalised Subdifferential at $\mathbf{x}_0$.

Definition 3.1.14: Generalised Jacobian Let a vector-valued function $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be given in component functions $\mathbf{F}(\mathbf{y}) = [f_1(\mathbf{y}), \dots, f_m(\mathbf{y})]$. Let $\bar{x}$ be a point around which each $f_i(\mathbf{y})$ is Lipschitz. Let $J\mathbf{F}(\mathbf{y})$ denote the usual $m \times n$ Jacobian matrix of partial derivatives whenever $\mathbf{y}$ is a point at which the necessary partial derivatives exist. The generalised Jacobian of $\mathbf{F}$ at $\bar{\mathbf{y}}$, denoted $\partial\mathbf{F}(\bar{\mathbf{y}})$ is the convex hull of all $m \times n$ matrices $\mathbf{Z}$ obtained as the limit of a sequence of the form $J\mathbf{F}(\mathbf{y}_i)$ where $\mathbf{y}_i \rightarrow \bar{\mathbf{y}}$ such that no $f_j(\mathbf{y}_i)$ fails to be differentiable at any $\mathbf{y}_i$.

$$\partial\mathbf{F}(\bar{\mathbf{y}}) = \text{co} \{ \lim J\mathbf{F}(\mathbf{y}_i) : \mathbf{y}_i \rightarrow \bar{\mathbf{y}}, \quad \mathbf{y}_i \notin \Omega_f \}, \quad (3.29)$$

where Ω_f denotes the set of $\mathbf{y}_i$ for which $\mathbf{F}$ fails to be differentiable.

Proposition 3.1.2

- $\partial \mathbf{F}(\mathbf{y})$ is a nonempty convex compact subset of $\mathbb{R}^{m \times n}$.
- $\partial \mathbf{F}(\mathbf{y})$ is closed at $\mathbf{y}$; that is, if $\mathbf{y}_i \rightarrow \mathbf{y}$, $\mathbf{Z}_i \in \partial \mathbf{F}(\mathbf{y}_i)$, $\mathbf{Z}_i \rightarrow \mathbf{Z}$, then $\mathbf{Z} \in \partial \mathbf{F}(\mathbf{y})$.
- $\partial \mathbf{F}(\mathbf{y})$ is upper semicontinuous at $\mathbf{y}$: for any $\epsilon > 0$ there is $\delta > 0$ such that, for all $\mathbf{y}^*$ in $\mathbf{y} + \delta \mathbb{B}$,

$$\partial \mathbf{F}(\mathbf{y}^*) \subset \partial \mathbf{F}(\mathbf{y}) + \epsilon \mathbb{B}_{m \times n}. \quad (3.30)$$

- If each component function f_i is Lipschitz of rank K_i at $\mathbf{y}$, then $\mathbf{F}$ is Lipschitz at $\mathbf{y}$ of rank $\mathbf{K} = [(K_1, K_2, \dots, K_m)]$, and $\partial \mathbf{F}(\mathbf{y}^*) \subset \partial \mathbf{F}(\mathbf{y}) + \mathbf{K} \mathbb{B}_{m \times n}$.
- $\partial \mathbf{F}(\mathbf{y}) \subset \partial f_1(\mathbf{y}) \times \partial f_2(\mathbf{y}) \times \dots \times \partial f_m(\mathbf{y})$ where the latter denotes the set of all matrices whose i th row belongs to $\partial f_i(\mathbf{y})$.

There are clear links between geometry and differentiation. The equivalence between analytical and geometrical concepts of differentiation are expressed in terms of the concepts of distance function and epigraph.

Definition 3.1.15 Let $\mathcal{S} \subset \mathbb{R}^n$ be a nonempty set. The distance function $d_{\mathcal{S}} : \mathbb{R}^n \rightarrow \mathbb{R}$ to the set $\mathcal{S}$ is defined by:

$$d_{\mathcal{S}}(\mathbf{x}) := \inf \{ \|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in \mathcal{S} \}, \quad \forall \mathbf{x} \in \mathbb{R}^n. \quad (3.31)$$

Theorem 3.1.2 The function $d_{\mathcal{S}}(\mathbf{x})$ is Lipschitz with constant $L = 1$:

$$|d_{\mathcal{S}}(\mathbf{x}) - d_{\mathcal{S}}(\mathbf{y})| \leq \|\mathbf{x} - \mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \quad (3.32)$$

Lemma 3.1.4 If the set $\mathcal{S}$ is convex then the function $d_{\mathcal{S}}(\mathbf{x})$ is also convex.

Lemma 3.1.5 If $\mathcal{S}$ is closed, then

$$\mathbf{x} \in \mathcal{S} \iff d_{\mathcal{S}}(\mathbf{x}) = 0. \quad (3.33)$$

Theorem 3.1.3 The tangent cone of the convex set $\mathcal{S}$ at $\mathbf{x} \in \mathcal{S}$ is defined by:

$$\mathcal{T}_{\mathcal{S}}(\mathbf{x}) = \{ \mathbf{y} \mid d'_{\mathcal{S}}(\mathbf{x}; \mathbf{y}) = 0 \}. \quad (3.34)$$

Theorem 3.1.4 The normal cone of the convex set $\mathcal{S}$ at $\mathbf{x} \in \mathcal{S}$ is defined by:

$$\mathcal{N}_{\mathcal{S}}(\mathbf{x}) = \text{cl} \left\{ \bigcup_{\lambda \geq 0} \partial d_{\mathcal{S}}(\mathbf{x}) \right\}. \quad (3.35)$$

Definition 3.1.16 The epigraph of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the following subset of $\mathbb{R}^n \times \mathbb{R}$:

$$\text{epi } f := \{ (\mathbf{y}, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(\mathbf{x}) \leq r \}. \quad (3.36)$$

Theorem 3.1.5 The epigraph of a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a closed convex subset of $\mathbb{R}^n \times \mathbb{R}$ and the epigraph of the function $\mathbf{d} \rightarrow f'(\mathbf{x}; \mathbf{d})$ is a convex cone containing zero.

The next two theorems show how tangents and normals are equivalently defined by using the epigraph of a convex function.

Theorem 3.1.6 If the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then

$$\text{epi } f'(\mathbf{x}; \cdot) = \mathcal{T}_{\text{epi } f}(\mathbf{x}, f(\mathbf{x})). \quad (3.37)$$

Theorem 3.1.7 If the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then

$$\partial f(\mathbf{x}) = \left\{ \boldsymbol{\xi} \in \mathbb{R}^n \mid (\boldsymbol{\xi}, -1) \in \mathcal{N}_{\text{epi } f}(\mathbf{x}, f(\mathbf{x})) \right\}. \quad (3.38)$$

The generalisations of the concepts of tangents and normals has been achieved by Clarke. In the nonconvex setting the tangent cone can be characterised by the distance function.

Definition 3.1.17 The tangent cone of the nonempty closed set $\mathcal{S}$ at $\mathbf{x} \in \mathcal{S}$ is defined by:

$$\mathcal{T}_{\mathcal{S}}(\mathbf{x}) = \{ \mathbf{y} \mid d_{\mathcal{S}}^0(\mathbf{x}; \mathbf{y}) \}. \quad (3.39)$$

Definition 3.1.18 The normal cone of the nonempty closed set $\mathcal{S}$ at $\mathbf{x} \in \mathcal{S}$ is defined by:

$$\mathcal{N}_{\mathcal{S}}(\mathbf{x}) := \mathcal{T}_{\mathcal{S}}(\mathbf{x})^\circ = \{ \mathbf{z} \in \mathbb{R}^n \mid \langle \mathbf{y}, \mathbf{z} \rangle \leq 0 \quad \forall \mathbf{y} \in \mathcal{T}_{\mathcal{S}}(\mathbf{x}) \}. \quad (3.40)$$

Theorem 3.1.8 If the set $\mathcal{S}$ is convex, then the tangent cone and normal cone of $\mathcal{S}$ as defined above coincide with the tangent cone and normal cone, respectively, defined earlier for convex sets.

Theorem 3.1.9 The tangent cone at $\mathbf{x}$ of the set $\mathcal{S}$ can also be written as

$$\mathcal{T}_{\mathcal{S}}(\mathbf{x}) := \{ \mathbf{y} \in \mathbb{R}^n \mid \forall t_j \downarrow 0 \quad \text{and} \quad \mathbf{x}_j \rightarrow \mathbf{x} \text{ with } \mathbf{x}_j \in \mathcal{S} \quad \exists \mathbf{y}_j \rightarrow \mathbf{y} \text{ with } \mathbf{x}_j + t_j \mathbf{y}_j \in \mathcal{S} \}. \quad (3.41)$$

Theorem 3.1.10 If $\mathbf{x} \in \text{int } \mathcal{S}$, then

$$\mathcal{T}_{\mathcal{S}}(\mathbf{x}) = \mathbb{R}^n \quad \text{and} \quad \mathcal{N}_{\mathcal{S}}(\mathbf{x}) = \mathbf{0}. \quad (3.42)$$

In the sequel the relations between epigraphs and level sets of nonsmooth functions is stated.

Theorem 3.1.11 If the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz at $\mathbf{x}$, then the epigraph of the function $\mathbf{d} \rightarrow f^\circ(\mathbf{x}, \mathbf{d})$ is a convex cone containing zero.

Theorem 3.1.12 If the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz at $\mathbf{x}$,

$$\text{epi } f^\circ(\mathbf{x}, \mathbf{d}) = \mathcal{T}_{\text{epi } f}(\mathbf{x}, f(\mathbf{x})). \quad (3.43)$$

Theorem 3.1.13 If the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz at $\mathbf{x}$, then

$$\partial f(\mathbf{x}) = \left\{ \boldsymbol{\xi} \in \mathbb{R}^n \mid (\boldsymbol{\xi}, -1) \in \mathcal{N}_{\text{epi } f}(\mathbf{x}, f(\mathbf{x})) \right\}. \quad (3.44)$$

Definition 3.1.19 The level set of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $\mathbf{x}$ is the set

$$\text{lev } f(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n \mid (\mathbf{y}, f(\mathbf{x})) \in \text{epi } f\}. \quad (3.45)$$

Theorem 3.1.14 Suppose that the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz at $\mathbf{x}$ and $\mathbf{0} \notin \partial f(\mathbf{x})$. Then

$$\{\mathbf{d} \in \mathbb{R}^n \mid (\mathbf{d}, 0) \in \text{epi } f^\circ(\mathbf{x}; \cdot)\} \supset \mathcal{T}_{\text{lev } f(\mathbf{x})}(\mathbf{x}). \quad (3.46)$$

If in addition f is regular at $\mathbf{x}$, then equality holds in (3.46).

Theorem 3.1.15 If the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz at $\mathbf{x}$ such that $\mathbf{0} \notin \partial f(\mathbf{x})$. Then

$$\mathcal{N}_{\text{lev } f(\mathbf{x})}(\mathbf{x}) \subset \bigcup_{\lambda \geq 0} \lambda \partial f(\mathbf{x}). \quad (3.47)$$

If, in addition, f is regular at $\mathbf{x}$, then equality holds in (3.47).

The relationship of various cones of nonsmooth analysis to proximal analysis establishes an important link to numerical methods.

Definition 3.1.20: Proximal Subgradient [31] A vector $\boldsymbol{\zeta} \in \mathcal{X}$ is called a proximal subgradient (or P-subgradient) of a lower semicontinuous function f at $\mathbf{x} \in \text{dom } f$ provided that

$$(\boldsymbol{\zeta}, -1) \in \mathcal{N}_{\text{epi } f}^{\text{P}}(\mathbf{x}, f(\mathbf{x})). \quad (3.48)$$

The set of all such $\boldsymbol{\zeta}$ is denoted $\partial^{\text{P}} f(\mathbf{x})$ and is referred to as the proximal subdifferential, or P-subdifferential.

Proposition 3.1.3

- A vector $\boldsymbol{\zeta}$ belongs to $\mathcal{N}_{\mathcal{S}}^{\text{P}}(\mathbf{s})$ if and only if there exists $\sigma = \sigma(\boldsymbol{\zeta}, \mathbf{s}) \geq 0$ such that

$$\langle \boldsymbol{\zeta}', \mathbf{s}' - \mathbf{s} \rangle \leq \sigma \|\mathbf{s}' - \mathbf{s}\|^2 \quad \forall \mathbf{s}' \in \mathcal{S}. \quad (3.49)$$

- Furthermore, for any given $\delta > 0$, one has $\boldsymbol{\zeta} \in \mathcal{N}_{\mathcal{S}}^{\text{P}}(\mathbf{s})$ if and only if there exists $\sigma = \sigma(\boldsymbol{\zeta}, \mathbf{s}) \geq 0$ such that

$$\langle \boldsymbol{\zeta}', \mathbf{s}' - \mathbf{s} \rangle \leq \sigma \|\mathbf{s}' - \mathbf{s}\|^2 \quad \forall \mathbf{s}' \in \mathcal{S}. \quad (3.50)$$

Definition 3.1.21: Proj Operator [31] Given the distance function $d_{\mathcal{S}}(\mathbf{x}) : \mathcal{X} \rightarrow \mathbb{R}$, which is stated by:

$$d_{\mathcal{S}}(\mathbf{x}) = \inf \{\|\mathbf{x} - \mathbf{s}\| : \mathbf{s} \in \mathcal{S}\}, \quad (3.51)$$

the set $\text{proj}_{\mathcal{S}}(\mathbf{x})$ consists of those points (if any) at which the infimum in (3.51) is attained:

$$\text{proj}_{\mathcal{S}}(\mathbf{x}) = \arg \min_{\mathbf{s} \in \mathcal{S}} d_{\mathcal{S}}(\mathbf{x}). \quad (3.52)$$

Proposition 3.1.4 Let $\mathcal{S}$ be a nonempty subset of $\mathcal{Y}$, and let $\mathbf{y} \in \mathcal{Y}$, $\mathbf{s} \in \mathcal{S}$. The following are equivalent:

- (a) $\mathbf{s} \in \text{proj}_{\mathcal{S}}(\mathbf{y})$;
- (b) $\mathbf{s} \in \text{proj}_{\mathcal{S}}(\mathbf{s} + t(\mathbf{y} - \mathbf{s})) \quad \forall t \in [0, 1]$;
- (c) $d_{\mathcal{S}}(\mathbf{s} + t(\mathbf{y} - \mathbf{s})) = t \|\mathbf{y} - \mathbf{s}\| \quad \forall t \in [0, 1]$;
- (d) $\langle \mathbf{y} - \mathbf{s}, \mathbf{s}' - \mathbf{s} \rangle \leq \frac{1}{2} \|\mathbf{s}' - \mathbf{s}\|^2 \quad \forall \mathbf{s}' \in \mathcal{S}$.

When $\mathcal{X}$ is finite-dimensional, to say that a sequence $\mathbf{x}_k$ converges to a point $\mathbf{x}$ means that $\|\mathbf{x}_k - \mathbf{x}\| \rightarrow 0$. This statement in turn means that for each $j = 1, 2, \dots, \dim(\mathcal{X})$, the j th component sequence $\langle \mathbf{e}_j, \mathbf{x}_k \rangle$ converges to the j th component of the limit vector $\langle \mathbf{e}_j, \mathbf{x} \rangle$. Here $\{\mathbf{e}_j\}$ is the standard orthonormal basis for $\mathcal{X}$. When $\mathcal{X}$ is not finite-dimensional, however, "componentwise" convergence is distinctly weaker than norm convergence.

Definition 3.1.22 Let $\mathcal{X}$ be a Hilbert space. A sequence $\mathbf{x}_k$ in $\mathcal{X}$ converges weakly to $\mathbf{x}$ if it satisfies

$$\lim_{i \rightarrow \infty} \langle \mathbf{p}, \mathbf{x}_i \rangle = \langle \mathbf{p}, \mathbf{x} \rangle \quad \forall \mathbf{p} \in \mathcal{X}. \quad (3.53)$$

An equivalent notation is $\mathbf{x} = \text{w-lim}_{i \rightarrow \infty} \mathbf{x}_i$.

Theorem 3.1.16 [31] let $\mathcal{X}$ be a Hilbert space.

- (a) If f is Lipschitz near $\mathbf{x}$, then

$$\partial f(\mathbf{x}) = \text{co} \left\{ \text{w-lim}_{i \rightarrow \infty} \zeta_i : \zeta_i \in \partial^P f(\mathbf{x}_i), \mathbf{x}_i \rightarrow \mathbf{x} \right\}. \quad (3.54)$$

- If $\mathcal{S}$ is a closed subset of $\mathcal{X}$ containing $\mathbf{x}$, then

$$\mathcal{N}_{\mathcal{S}}(\mathbf{x}) = \text{co} \left\{ \text{w-lim}_{i \rightarrow \infty} \zeta_i : \zeta_i \in \mathcal{N}_{\mathcal{S}}^P(\mathbf{x}_i), \mathbf{x}_i \rightarrow \mathbf{x} \right\}. \quad (3.55)$$

Lemma 3.1.6 Consider a symmetric matrix $\mathbf{P}$ and a positive semi-definite matrix $\mathbf{Q}$. For each $\mathbf{x}$ that is not in the nullspace of $\mathbf{Q}$, one has $\langle \mathbf{x}, \mathbf{Q} \mathbf{x} \rangle > 0$, further for sufficiently large c one has $\langle \mathbf{x}, (\mathbf{P} + c\mathbf{Q}) \mathbf{x} \rangle > 0$.

Given a nonlinear unconstrained optimisation problem of the form

$$\min f(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^n, \quad (3.56)$$

where the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is supposed to be locally Lipschitz continuous for all $\mathbf{x} \in \mathbb{R}^n$. If f is continuously differentiable, then problem (3.56) is said to be smooth, otherwise it is classified as nonsmooth. A point $\mathbf{x} \in \mathbb{R}^n$ is a global minimum of f if it satisfies $f(\mathbf{x}) \leq f(\mathbf{y})$, $\forall \mathbf{y} \in \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is a local minimum of f if there exists $\epsilon > 0$ such that $f(\mathbf{x}) \leq f(\mathbf{y})$, $\forall \mathbf{y} \in \mathbb{B}(\mathbf{x}, \epsilon)$. The necessary conditions for a locally Lipschitz continuous function to attain its local minimum in an unconstrained case are given in the following:

Theorem 3.1.17 Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function at $\mathbf{x} \in \mathbb{R}^n$. If f attains its local minimal value at $\mathbf{x}$, then

1. $\mathbf{0} \in \partial f(\mathbf{x})$,
2. $f^\circ(\mathbf{x}; \mathbf{d}) \geq 0, \forall \mathbf{d} \in \mathbb{R}^n$.

Theorem 3.1.18 If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function, then the following conditions are equivalent:

1. Function f attains its global minimal value at $\mathbf{x}$,
2. $\mathbf{0} \in \partial f(\mathbf{x})$,
3. $f'(\mathbf{x}; \mathbf{d}) \geq 0, \forall \mathbf{d} \in \mathbb{R}^n$.

A point $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{0} \in \partial f(\mathbf{x})$ is called a stationary point of f .

3.1.1 Complementarity Problems and Variational Inequalities

The simplest and most widely studied of the complementarity problems is the LCP, because of its relevance to the quadratic programming (QP) problem with inequality constraints. In references [32] by Cottle et, al. and [70] by Murty detailed treatment of complementarities and optimisation can be found. It is well-known, that the first and second-order necessary optimality conditions for QP with inequality constraints in nonnegative variables are representable as LCP. The first-order necessary optimality conditions for a quadratic program involving some equality and some inequality constraints are in this form.

Definition 3.1.23: The Nonlinear Complementarity Problem Given a mapping $\mathbf{F}(\mathbf{z}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, find a $\mathbf{z} \in \mathbb{R}^n$ satisfying

$$\mathbf{z} \succeq \mathbf{0}, \quad \mathbf{F}(\mathbf{z}) \succeq \mathbf{0}, \quad \langle \mathbf{z}, \mathbf{F}(\mathbf{z}) \rangle = 0. \quad (3.57)$$

If $\mathbf{F}(\mathbf{z})$ is the affine function $\mathbf{b} + \mathbf{A}\mathbf{z}$, then the standart NCP becomes the LCP $(\mathbf{b}, \mathbf{A})$. A further generalisation of the NCP is the VIP:

Definition 3.1.24: The Variational Inequality Problem [70] Given a mapping $\mathbf{F}(\mathbf{z}) = \{F_i(\mathbf{z})\} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $\mathcal{K} \subset \mathbb{R}^n$, $\mathcal{K} \neq \emptyset$, find a $\mathbf{z}^* \in \mathcal{K}$ satisfying

$$\langle \mathbf{y} - \mathbf{z}^*, \mathbf{F}(\mathbf{z}^*) \rangle \geq 0, \quad \forall \mathbf{y} \in \mathcal{K}. \quad (3.58)$$

This problem is denoted by $VI(\mathcal{K}, \mathbf{F})$. Some important classifications of the VI based on the structure of the set $\mathcal{K}$ and its relation to various complementarity problems is established as follows:

- If $\mathcal{K} = \{\mathbf{z} \mid \mathbf{z} \in \mathbb{R}_0^n\}$, then the $\mathbf{z}^*$ solving (3.58) also solves (3.57).
- If $\mathcal{K}$ is polyhedral and $\mathbf{F}$ is affine, it can be verified that $VI(\mathcal{K}, \mathbf{F})$ is an LCP.

- When $\mathcal{K}$ is a rectangular region, this is called the Box Constrained VIP (BVIP), which is also commonly referred to as the (nonlinear) Mixed Complementarity Problem (MCP).
- For any subset $\mathcal{K} \subset \mathbb{R}^n$, its polar cone, denoted by $\mathcal{K}^\circ$, is defined by

$$\mathcal{K}^\circ = \{\mathbf{y} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle \geq 0, \quad \forall \mathbf{x} \in \mathcal{K}\}. \quad (3.59)$$

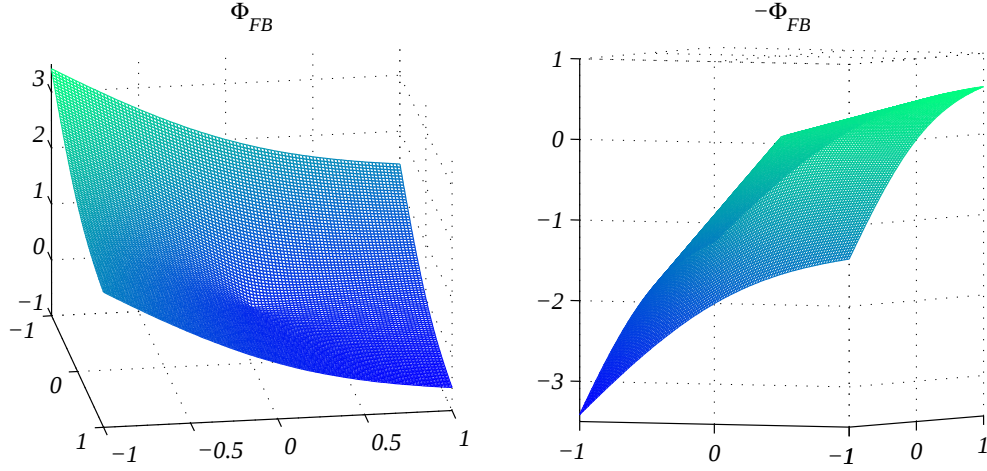
The complexity of the numerical behaviour of nonlinear programming problems with complementarity constraints vary strongly depending on the structure of the linear operator $\mathbf{A}$. It has been demonstrated that the general LCP $(\mathbf{b}, \mathbf{A})$ with general integer data is NP-hard. The only known algorithms that are guaranteed to process a general LCP $(\mathbf{b}, \mathbf{A})$ with no restrictions on the data are enumerative algorithms. Among the first class of methods which were developed for the LCP are the pivotal methods which try to obtain a basic feasible complementary vector through a series of pivot steps. These methods are variants of the complementary pivot method of Lemke and Howson [55]. The most famous among the pivotal methods for the LCP is Lemke's method. The other important class of methods for the LCP are the interior point methods. Interior point methods originated from an algorithm introduced by Karmarkar in 1984 for solving linear problems. The most successful interior point methods follow a path in $\mathcal{F}^0 = \{(\mathbf{w}, \mathbf{z}) \mid \mathbf{w} - \mathbf{A}\mathbf{z} = \mathbf{b}, \mathbf{w} \succ 0, \mathbf{z} \succ 0\}$, in an effort to reduce $\langle \mathbf{w}, \mathbf{z} \rangle$ to 0. One such typical method defines this path as the set of solutions to the following parameterised system

$$\mathbf{w} - \mathbf{A}\mathbf{z} = \mathbf{b}, \quad (3.60)$$

$$\langle \mathbf{w}_i, \mathbf{z}_i \rangle = \mu, \quad i = 1, \dots, N, \quad (3.61)$$

$$\mathbf{w} \succeq \mathbf{0}, \quad \mathbf{z} \succeq \mathbf{0}. \quad (3.62)$$

where each choice of the parameter μ yields a different point along the path. Interior point methods for convex QP essentially use the above methods on the KKT conditions, which form an LCP or mLCP. These methods have polynomial time worst case complexity for monotone complementarity problems, which correspond to convex quadratic programs (convex QPs). Solving nonlinear programming problems with general nonlinear complementarity constraints (NCP) has naturally a higher-degree of sophistication. The first methods considered for NCP's has been sequential LCP methods. These methods generate a sequence of iterates $\{\mathbf{z}^k\}$, such that $\{\mathbf{z}^{k+1}\}$ is a solution to a linear complementarity problem $(\mathbf{b}^k, \mathbf{A}^k)$, where $\mathbf{b}^k$ and $\mathbf{A}^k$ are chosen to approximate $\mathbf{F}$ near $\mathbf{x}^k$. Depending on the choice of $\mathbf{A}^k$ and $\mathbf{b}^k$ various algorithms can be generated, each of which is analogous to a standard iterative method for solving nonlinear systems of equations. Among these are Newton, quasi-Newton, Jacobi, successive overrelaxation, symmetrised Newton and projection methods. Details how to choose $\mathbf{A}^k$ and $\mathbf{b}^k$ for each of these methods can be found in Fukushima *et al.* [39]. The choices that result in Newton's Method are $\mathbf{A}^k = \nabla \mathbf{F}(\mathbf{z}^k)$ and $\mathbf{b}^k = \mathbf{F}(\mathbf{z}^k) - \mathbf{A}^k \mathbf{z}^k$. It has been shown that, in a neighbourhood

Figure 3.2: The functions Φ_{FB} and $-\Phi_{FB}$.

of a solution $\mathbf{z}^*$ to the NCP, the iterates produced by this method are well-defined and converge quadratically to $\mathbf{z}^*$ provided that $\nabla \mathbf{F}$ is locally Lipschitzian at $\mathbf{z}^*$ and that a certain strong regularity assumption is satisfied. Another class of methods for NCP involves reformulating the problem as a system of nonlinear equations. This involves constructing a function $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the property that zeros of H correspond to solutions of the NCP. Such a function H is called an NCP-function.

Many other NCP functions have been studied in the literature. Interestingly, while smooth NCP functions exist, they are generally not in favor computationally since they have singular Jacobian matrices at degenerate solutions. A degenerate solution is a point $\mathbf{z}^*$ such that $F_i(\mathbf{z}^*)$ and $\mathbf{z}_i^*$ are both zero for some index i . Thus, the NCP functions of interest are usually only piecewise differentiable. There are several NCP functions which are developed already:

- The Fischer-Burmeister function is given by

$$\Phi_{FB}(x, y) = x + y - \sqrt{x^2 + y^2}, \quad (3.63)$$

which has first been introduced in [38] for nonlinear programming. It is nondifferentiable at the origin, and its Hessian is unbounded at the origin. By considering the Fenchel conjugate of the Fischer-Burmeister function, the subgradient can be analysed:

$$f^*(m_x, m_y) = \sup_{x, y} \{m_x x + m_y y + \sqrt{x^2 + y^2} - x - y\} \quad (3.64)$$

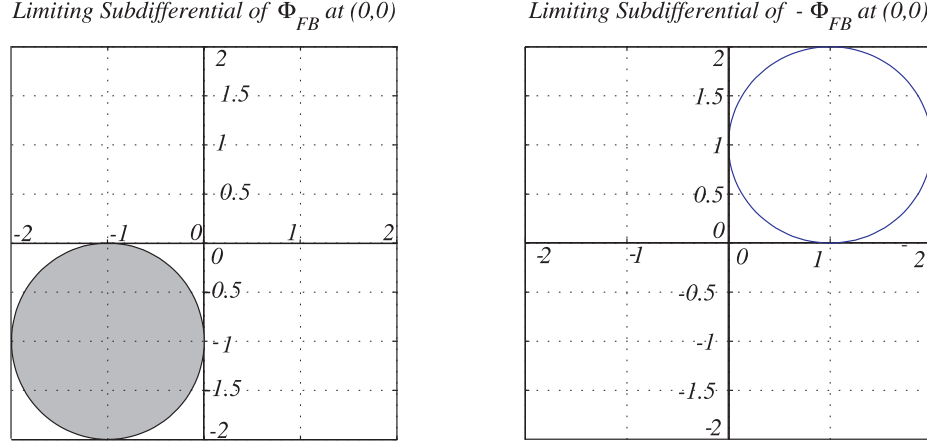


Figure 3.3: The limiting subdifferentials of Φ_{FB} and $-\Phi_{FB}$ at the origin.

The conjugate function attains its maximum if following relations on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ are fulfilled:

$$\begin{aligned} (m_x + 1) - \frac{x}{\sqrt{x^2 + y^2}} &= 0, \\ (m_y + 1) - \frac{y}{\sqrt{x^2 + y^2}} &= 0. \end{aligned}$$

The domain of definition of the relations is extended by eliminating the term $\sqrt{x^2 + y^2}$ in the denominators as:

$$(m_x + 1)^2 + (m_y + 1)^2 = 1.$$

Except at the origin, the gradient of the Fischer-Burmeister function is an element of the unitary circle with its center at $(-1, -1)$ and is single-valued. The function has no stationary points or extrema since both m_x and m_y can not become zero. At the origin, the subdifferential of this convex function consists of the unitary disc with its origin at $(-1, -1)$. Since the origin is not an element of the unitary disc the origin is neither a stationary point nor an extrema. So the Fischer-Burmeister function is a convex function without any extrema on $\mathbb{R}^2$. On the other, if one considers the set of proximal subgradients of $-\Phi_{FB}(x, y)$ at the origin, one sees that $\mathcal{N}_{\text{epi}-\Phi_{FB}}^P$ is given by:

$$\mathcal{N}_{\text{epi}-\Phi_{FB}}^P = \left\{ \begin{pmatrix} \boldsymbol{\xi} \\ -1 \end{pmatrix} \left\| \boldsymbol{\xi} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = 1 \right\}, \quad (3.65)$$

which is a nonconvex set. It describes a unit circle with unit radius and center $(1, 1)$.

- The min-function is the nonsmooth function

$$\Phi_{\min}(x, y) = \min(x, y). \quad (3.66)$$

It can be written equivalently in terms of the natural residual function

$$\Phi_{NR}(x, y) = \frac{1}{2} \left(x + y - \sqrt{(x - y)^2} \right). \quad (3.67)$$

This function is again nondifferentiable at the origin and along the line $x = y$.

- The Chen-Chen-Kanzow function is a convex combination of the Fischer-Burmeister function and the bilinear function xy . For a fixed parameter $\lambda \in (0, 1)$, it is defined as

$$\Phi_{CCK}(x, y) = \lambda \Phi_{FB}(x, y) + (1 - \lambda)x_+ y_+, \quad (3.68)$$

where $x_+ = \max(0, x)$. Note that for $a \geq 0$, $a_+ = a$; hence, for any method that remains feasible with respect to the simple bounds

$$\Phi_{CCK}(x, y) = \lambda \Phi_{FB}(x, y) + (1 - \lambda)xy. \quad (3.69)$$

An alternative approach to applying a generalised Newton method is to approximate the non-smooth system by a family of smoothing functions. This is the fundamental idea behind the so-called smoothing methods. The basic idea of these techniques is to approximate the function H by a family of smooth approximations H_μ parameterised by the scalar μ . Two typical smoothing functions for Φ_{FB} are

$$\Phi_{NRs1}(x, y) = \frac{1}{2} \left(x + y - \sqrt{(x - y)^2 + 4\mu^2} \right) \quad (3.70)$$

and

$$\Phi_{NRs2}(x, y) = \frac{1}{2} \left(x + y - \sqrt{(x - y)^2 + \frac{xy}{\sigma}} \right). \quad (3.71)$$

In (3.70) the complementarity is satisfied only up to μ^2 at the solution. In contrast, smoothing formulation (3.71) enables that the smoothing remains exact in the sense that $\Phi_{NRs}(x, y) = 0$ if and only if $0 \leq x \perp y \leq 0$. Another class of algorithms for monotone NCP are interior point algorithms. In similar fashion to the interior point techniques for LCP, the methods follow the central path defined by:

$$\mathbf{w} = \mathbf{F}(\mathbf{z}), \quad (\mathbf{w}, \mathbf{z}) \succeq \mathbf{0}, \quad \mathbf{w}_i \mathbf{z}_i = \mu \quad (3.72)$$

which leads to a solution as $\mu \downarrow 0$. The final class of methods are continuation or homotopy methods. Like the smoothing and interior point algorithms, these methods work by introducing an additional variable μ and then following a path which leads to a solution. However, unlike the smoothing and interior point methods, the continuation methods do not assume that μ decreases monotonically along this path.

3.2 Global and Local Properties of the Augmented Lagrangian Method

Lagrangian is related to the optimisation techniques called quadratic penalty methods. In addition to the quadratic penalty terms it also has the linear terms due to the Lagrangian formulation. In the works of Rockafellar [80], [81], [86] an overview on the properties and the need for the development of augmented Lagrangian approach is expounded. In [14] Bertsekas gives a sound introduction to primal-dual methods related to augmented Lagrangians. Its main advantage in comparison to the pure quadratic penalty methods is that ill-conditioning due to high penalty parameters is alleviated. Ill-conditioning can be visualised as narrow level sets in some directions and numerically can be characterised by a very high condition number if the nonlinear optimisation problem possesses sufficient differentiability properties. It can be shown that the method converges for reasonably high penalty parameters to the optimal primal and dual values exactly by providing external estimates to the dual multipliers at each successive minimisation step. Further, the infeasibility with respects to the initial constraints are eliminated, by the penalisation method. The penalisation of the constraints quadratically with a suitably high penalty parameter induces a local convexification of the feasible set by suitable conditions that is shown by a generalised Hessian analysis. The analysis is carried out in two parts. First, the local properties of the augmented Lagrangian function and its relation to the usual Lagrangian is investigated. In the second part, the global properties of the augmented Lagrangian is treated. The value function is introduced, in order to discuss the numerical sensitivity to some parameters of the underlying optimisation problem. As a known fact, the dual variables if they exist, provide means for the sensitivity of the overall optimisation problem to each constraint. The main tool in the analysis of local properties of the augmented Lagrangian function is the consideration of the proximal subdifferential inequality to the value function resulting from the perturbations of the inequality and equality constraints. The resemblance in structure of the augmented Lagrangian function to the minimisation problem that characterises the proximal subdifferential inequality is exploited in general normed spaces for this analysis. In the sequel, the some fundamental theorems and their properties in convex analysis, non-smooth analysis and differentiation in finite-dimensional spaces is stated without proof. The interested reader is referred to [78] for convex analysis, to [29] for nonconvex analysis and to [31] for projections to closed set in finite-dimensional spaces.

3.2.1 The Lagrangian and the Augmented Lagrangian

Given a Banach space $\mathcal{Y}$ and its dual space $\mathcal{Y}^*$, the general nonlinear programming problem P is to minimize a function $f(\mathbf{y})$ on $\mathcal{Y}$ subject to equality constraints $\mathbf{h}_j(\mathbf{y}) = 0$, $j = 1, 2, \dots, m$,

inequality constraints $\mathbf{l}_i(\mathbf{y}) \leq 0$, $i = 1, 2, \dots, r$ is stated as:

$$\min f(\mathbf{y}) \quad (3.73)$$

$$\text{subject to} \quad h_1(\mathbf{y}) = 0, \dots, h_m(\mathbf{y}) = 0, \quad (3.74)$$

$$\mathbf{l}_1(\mathbf{y}) \leq 0, \dots, \mathbf{l}_r(\mathbf{y}) \leq 0. \quad (3.75)$$

The equality and inequality constraints can be considered as a special abstract constraint, of which the structure is useful for further analysis. Here it is assumed that $\mathcal{C}$ is closed, and the functions $\mathbf{f}, \mathbf{g}_i, \mathbf{h}_j$ are Lipschitz continuous. The Lagrangian of this problem is given by:

$$\mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \kappa) = \kappa f(\mathbf{y}) + \langle \boldsymbol{\lambda}, \mathbf{h}(\mathbf{y}) \rangle + \langle \boldsymbol{\mu}, \mathbf{g}(\mathbf{y}) \rangle. \quad (3.76)$$

The vectors $\boldsymbol{\lambda} \in \mathcal{Y}^*$ and $\boldsymbol{\mu} \in \mathcal{Y}^*$ are dual multipliers. The parameter κ is a parameter. Making use of proposition 3.1.4, one can state following alternative forms which are equivalent to each other:

Corollary 3.2.1

- (a) $\mathbf{g}_i(\mathbf{y}) = \text{proj}_{\mathbb{R}_0^+}(\mathbf{l}_i(\mathbf{y}))$;
- (b) $\mathbf{g}_i(\mathbf{y}) = \text{proj}_{\mathbb{R}_0^+}(\mathbf{g}_i(\mathbf{y}) + t(\mathbf{l}_i(\mathbf{y}) - \mathbf{g}_i(\mathbf{y}))) \quad \forall t \in [0, 1]$;
- (c) $d_{\mathbb{R}_0^+}(\mathbf{g}_i(\mathbf{y}) + t(\mathbf{l}_i(\mathbf{y}) - \mathbf{g}_i(\mathbf{y}))) = t \|\mathbf{l}_i(\mathbf{y}) - \mathbf{g}_i(\mathbf{y})\| \quad \forall t \in [0, 1]$;
- (d) $\langle \mathbf{g}_i(\mathbf{y}) - \mathbf{l}_i(\mathbf{y}), \mathbf{g}_i(\mathbf{y}') - \mathbf{g}_i(\mathbf{y}) \rangle \leq \frac{1}{2} \|\mathbf{g}_i(\mathbf{y}') - \mathbf{g}_i(\mathbf{y})\|^2 \quad \forall \mathbf{g}_i(\mathbf{y}') \in \mathbb{R}_0^+$.

An inequality constrained of the form $\mathbf{l}_i(\mathbf{y}) \leq 0$ can equivalently be converted into an equality constrained as follows:

$$\mathbf{g}_i(\mathbf{y}) = \max(0, \mathbf{l}_i(\mathbf{y})), \quad (3.77)$$

which is equivalent to:

$$\mathbf{g}_i(\mathbf{y}) = \text{proj}_{\mathbb{R}_0^+}(\mathbf{l}_i(\mathbf{y})). \quad (3.78)$$

The augmented Lagrangian problem considers a modification of the above problem P and is denoted by P_A :

$$\min \kappa f(\mathbf{y}) + \frac{1}{2}c \sum_{i=0}^m h_i^2(\mathbf{y}) + \frac{1}{2}c \sum_{j=0}^r \max\{0, \mathbf{l}_j(\mathbf{y})\}^2, \quad (3.79)$$

$$\text{subject to} \quad h_1(\mathbf{y}) = 0, \dots, h_m(\mathbf{y}) = 0, \quad (3.80)$$

$$\mathbf{l}_1(\mathbf{y}) \leq 0, \dots, \mathbf{l}_r(\mathbf{y}) \leq 0. \quad (3.81)$$

Here the parameter c is the penalty parameter. The augmented Lagrangian is the Lagrangian to the problem P_A :

$$\mathcal{L}_a = \kappa f(\mathbf{y}) + \langle \bar{\boldsymbol{\lambda}}, \mathbf{h}(\mathbf{y}) \rangle + \langle \bar{\boldsymbol{\mu}}, \mathbf{g}(\mathbf{y}) \rangle + \frac{1}{2}c \left(\sum_{i=0}^m h_i^2(\mathbf{y}) + \sum_{j=0}^r \max\{0, \mathbf{l}_j(\mathbf{y})\}^2 \right). \quad (3.82)$$

The Lagrangian of P_A can after some algebraical manipulations equivalently be rewritten as:

$$\mathcal{L}_a = \kappa f(\mathbf{y}) + \langle \bar{\boldsymbol{\lambda}}, \mathbf{h}(\mathbf{y}) \rangle + \frac{1}{2}c \left(\sum_{i=0}^m h_i^2(\mathbf{y}) + \frac{1}{c^2} \sum_{j=0}^r (\max\{0, \bar{\boldsymbol{\mu}}_j + c l_j(\mathbf{y})\}^2 - \bar{\boldsymbol{\mu}}_j^2) \right). \quad (3.83)$$

The associated Lagrange multipliers of the problem P_A are denoted by $\bar{\boldsymbol{\mu}}$ and $\bar{\boldsymbol{\lambda}}$. The quadratic penalty function method consists of solving a sequence of problems so that both approaches mentioned above are utilised together in order to obtain the saddle point of the Lagrangian. Following sequence of problems are being solved:

$$\min \mathcal{L}_{a^k}(\mathbf{y}, \boldsymbol{\lambda}^k) \quad (3.84)$$

$$\text{subject to } \mathbf{y} \in \mathcal{Y} \quad (3.85)$$

where $\{\boldsymbol{\lambda}^k\}$ is a sequence in $\mathbb{R}^m$ and $\{c^k\}$ is a positive penalty parameter sequence that increases to ∞ .

Theorem 3.2.1: Lagrange Multiplier Rule [29] Let $\mathbf{y}$ solve P , there exist $\boldsymbol{\mu} \geq \mathbf{0}$, $\boldsymbol{\mu} \in \mathcal{Y}^*$, $\boldsymbol{\lambda} \in \mathcal{Y}^*$ not all zero, such that $\boldsymbol{\lambda}_i \mathbf{h}_i(\mathbf{y}) = 0$, for $i = 1, \dots, m$; $\boldsymbol{\mu}_j \mathbf{g}_j(\mathbf{y}) \leq 0$, for $j = 1, \dots, r$ and $\mathbf{0} \in \partial_{\mathbf{y}} \mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \kappa)$.

Lemma 3.2.1 Let $\mathbf{y}$ be feasible for P and let κ be nonnegative. If $(\kappa, \boldsymbol{\lambda}, \boldsymbol{\beta})$ satisfy the Lagrange multiplier rule, so does also the scaled $(1, \frac{\boldsymbol{\lambda}}{\kappa}, \frac{\boldsymbol{\beta}}{\kappa})$, for $\kappa \neq 0$.

The dual problem to the problem P is the maximisation of a concave function of the Lagrange multipliers. This corresponds to the dual problem D :

$$\max_{\boldsymbol{\lambda}, \boldsymbol{\mu}} g(\boldsymbol{\lambda}, \boldsymbol{\mu}) \quad (3.86)$$

$$\boldsymbol{\lambda} \in \mathbb{R}^m, \quad \boldsymbol{\mu} \in {}^+\mathbb{R}_0^r \quad (3.87)$$

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\mathbf{y} \in \mathcal{Y}} \mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \kappa). \quad (3.88)$$

The optimal values in these two problems satisfy

$$\inf (P) \geq \sup (D), \quad (3.89)$$

or

$$\inf_{\mathbf{y}} \sup_{\boldsymbol{\lambda}, \boldsymbol{\mu}} \mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \kappa) \geq \sup_{\boldsymbol{\lambda}, \boldsymbol{\mu}} \inf_{\mathbf{y}} \mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \kappa). \quad (3.90)$$

In nonlinear programming there is generally a duality gap between the primal and dual problems if the optimisation problem is considered in terms its ordinary Lagrangian, unless the problem is convex. If the augmented Lagrangian to a nonlinear programming problem is considered then the duality gap is removable, which is generally and mostly obtained by adding quadratic penalty-like terms. The dual problem to the problem P_A is the maximisation of a concave

function of the Lagrange multipliers and an penalty parameter. This corresponds to the dual problem $D_{\mathcal{A}}$:

$$\max_{\lambda, \mu} g_{\mathcal{A}}(\lambda, \mu) \quad (3.91)$$

$$\lambda \in \mathbb{R}^m, \quad \mu \in {}^+\mathbb{R}_0^r \quad (3.92)$$

$$g_{\mathcal{A}}(\lambda, \mu) = \inf_{y \in \mathcal{Y}} \mathcal{L}_{\mathcal{A}}(y, \lambda, \mu, \kappa) \quad (3.93)$$

For the augmented Lagrangian problem then following is valid:

$$\inf_y \sup_{\lambda, \mu} \mathcal{L}_{\mathcal{A}}(y, \lambda, \mu, \kappa) = \sup_{\lambda, \mu} \inf_y \mathcal{L}_{\mathcal{A}}(y, \lambda, \mu, \kappa). \quad (3.94)$$

3.2.2 Normality and Degeneracy of the NLP Problem

In the sequel it is assumed that f , $\mathbf{h}$ and $\mathbf{g}$ are locally Lipschitz and a growth hypotheses holds. Throughout the analysis in the sequel the growth hypothesis is valid, which is stated below:

Hypothesis 3.2.1: Growth Hypothesis For every $s, t \in \mathbb{R}$ and $q \in \mathbb{R}^r$, the following set is bounded:

$$\{y \in \mathbb{R}^n \mid f(y) \leq t, \|\mathbf{h}(y)\| \leq s, \mathbf{g}(y) \leq q\}. \quad (3.95)$$

The set of Lagrange multipliers $\mathcal{M}^k$ for problem P is the set of those λ and μ for which:

$$\mathcal{M}^k(y) = \{ \{\lambda, \mu\} \mid \mathbf{0} \in \partial \{ \kappa f + \langle \lambda, \mathbf{h}(y) \rangle + \langle \mu, \mathbf{g}(y) \rangle \} \} \quad (3.96)$$

is valid. The problem is called normal if:

$$\mathbf{0} \in \partial \{ \langle \lambda, \mathbf{h}(y) \rangle + \langle \mu, \mathbf{g}(y) \rangle \} \quad (3.97)$$

is only valid, when $\lambda = \mathbf{0}$, $\mu = \mathbf{0}$ for all y . In the case, where the equality constraints are C^1 , this is equivalent to condition that the jacobian of the equality constraints have full rank. Analogously the set of dual multipliers of problem $P_{\mathcal{A}}$, $\mathcal{M}_{\mathcal{A}}^k(y)$ consists of multipliers $\bar{\alpha}$ and $\bar{\mu}$. The problem P or $P_{\mathcal{A}}$ is called degenerate or abnormal when the multiplier κ corresponding to f becomes 0. Various normality or constrained qualification conditions are proposed in order guarantee that the necessary conditions are valid with $\kappa = 1$. In the sequel these constraint qualification conditions are discussed. It is possible to classify constraint qualifications into two categories. In the first category, there are constraint qualifications that make structural assumptions about the data of the problem so that the set $\mathcal{M}^0(y)$ of abnormal multipliers necessarily reduces to $\mathbf{0}$, such as the Mangasarian-Fromowitz or Slater constraint qualifications. In the second category falls the constraint qualification called calmness that assures that $\mathcal{M}^1(y)$ is nonempty even if $\mathcal{M}^0(y)$ does not reduce to zero. This classification enables an alternative definition of the Lagrange multiplier rule:

Theorem 3.2.2: Lagrange Multiplier Rule II [29] If $\mathbf{y}$ solves problem P , then one has

$$\mathcal{M}^1(\mathbf{y}) \cup [\mathcal{M}^0(\mathbf{y}) \setminus \{\mathbf{0}\}] \neq \emptyset. \quad (3.98)$$

By the fact that the normality conditions of problems P and $P_{\mathcal{A}}$ are equivalent, it is easily deduced that $\mathcal{M}^0 \neq \mathbf{0}$, if and only if $\mathcal{M}_{\mathcal{A}}^0 \neq \mathbf{0}$.

The Mangasarian-Fromowitz conditions postulate that if the functions $\mathbf{g}_i$, $\mathbf{h}_j$ are continuously differentiable and $\mathcal{C} = \mathcal{Y}$, that the vectors $\nabla \mathbf{h}_j$, $j = 1, 2, \dots, m$ are linearly independent, and that there is a vector $\mathbf{v}$ such that

$$\langle \nabla \mathbf{h}_j, \mathbf{v} \rangle = 0, j = 1, 2, \dots, m \quad (3.99)$$

$$\langle \nabla \mathbf{g}_i, \mathbf{v} \rangle < 0 \quad \text{if} \quad \mathbf{g}_i(\mathbf{y}) = 0, i = 1, 2, \dots, n. \quad (3.100)$$

The Slater condition states that if problem P has no equality constraints and the inequality constraints $\mathbf{g}_i$, and $\mathcal{C}$ are convex, the problem P is called strictly feasible if there exists a point $\bar{\mathbf{y}}$ in $\mathcal{C}$ such that $g_i(\bar{\mathbf{y}}) < 0, i = 1, 2, \dots, n$. If the strict feasibility condition is valid as above, then the problem P is normal or nondegenerate.

Let $P(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the problem of minimizing $f(\mathbf{y})$ subject to the perturbed constraints $\mathbf{h}(\mathbf{y}) + \boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{g}(\mathbf{y}) + \boldsymbol{\beta} = \mathbf{0}$, and let $V(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be the corresponding value function:

$$V(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \inf_{\mathbf{y}} \{f(\mathbf{y}) \mid \mathbf{h}(\mathbf{y}) + \boldsymbol{\alpha} = \mathbf{0}, \mathbf{g}(\mathbf{y}) + \boldsymbol{\beta} = \mathbf{0}\}. \quad (3.101)$$

V may take values in the range $[-\infty, +\infty]$, the value $+\infty$ corresponding to the cases in which the feasible set for $P(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is empty. The feasible set for $P(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the set:

$$\Phi(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{h}(\mathbf{y}) + \boldsymbol{\alpha} = \mathbf{0}, \mathbf{g}(\mathbf{y}) + \boldsymbol{\beta} = \mathbf{0}\}. \quad (3.102)$$

Further let $\Sigma(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the set of $\mathbf{y}$ for which the infimum of $P(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is attained:

$$\Sigma(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \arg \min \{f(\mathbf{y}) \mid \mathbf{h}(\mathbf{y}) + \boldsymbol{\alpha} = \mathbf{0}, \mathbf{g}(\mathbf{y}) + \boldsymbol{\beta} = \mathbf{0}\}. \quad (3.103)$$

The value function $V_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for problem $P_{\mathcal{A}}$ is analogously given by:

$$V_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \inf_{\mathbf{y}} \{P_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\}. \quad (3.104)$$

Similarly, the set $\Sigma_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and the set $\Phi_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ can be defined as:

$$\Sigma_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \arg \min_{\mathbf{y}} \{P_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\} \quad (3.105)$$

and

$$\Phi_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \{\mathbf{y} \in \mathbb{R}^n \mid \inf_{\mathbf{y}} \{P_{\mathcal{A}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\} < +\infty\}. \quad (3.106)$$

For sufficiently high penalty parameters c , following is valid:

$$\Phi_{\mathcal{A}}(\alpha, \beta) = \Phi(\alpha, \beta), \quad (3.107)$$

$$\Sigma_{\mathcal{A}}(\alpha, \beta) = \Sigma(\alpha, \beta). \quad (3.108)$$

Definition 3.2.1

At $\mathbf{y}$ that solves P , the problem P is calm at $\mathbf{y}$ provided there exist positive ϵ and L such that, $\forall (\alpha, \beta) \in \epsilon \mathbb{B}$, and $\bar{\mathbf{y}}$ in $\mathbf{y} + \epsilon \mathbb{B}$ which are feasible for $P(\alpha, \beta)$, one has

$$f(\bar{\mathbf{y}}) - f(\mathbf{y}) + L |(\alpha, \beta)| \geq 0. \quad (3.109)$$

Proposition 3.2.1 Let $V(\mathbf{0}, \mathbf{0})$ be finite, and suppose that one has

$$\liminf_{(\alpha, \beta) \rightarrow (\mathbf{0}, \mathbf{0})} \frac{V(\alpha, \beta) - V(\mathbf{0}, \mathbf{0})}{|(\alpha, \beta)|} > -\infty, \quad (3.110)$$

then, for any solution $\mathbf{y}$ to P , P is calm at $\mathbf{y}$.

One difficulty in characterizing $\partial V(\alpha, \beta)$ in terms of proximal subgradients is that such limits may fail to exist in the ordinary sense due to unboundedness. Due to this problem in [86] an extended characterisation of the subdifferential set composed of points and directions is suggested. These directions are rays emanating from the origin which represent in some sense points in infinity. Then it becomes possible to represent the subdifferential as a convex hull of points and directions. The following theorem shall in this terms help to characterize the limiting subdifferential in terms convex hulls of points and rays.

Theorem 3.2.3 [86] Let $V(\alpha, \beta) \rightarrow [-\infty, +\infty]$ be lower semicontinuous, and let $(\hat{\alpha}, \hat{\beta})$ be any point with $V(\hat{\alpha}, \hat{\beta})$ finite. Define

$$\begin{aligned} \partial^P V(\hat{\alpha}, \hat{\beta}) = & \left\{ (\hat{\lambda}, \hat{\mu}) \in \mathbb{R}^m \times \mathbb{R}^r \mid \exists (\hat{\alpha}^k, \hat{\beta}^k) \text{ proximal subgradient of } V(\hat{\alpha}, \hat{\beta}) \right. \\ & \left. \text{at } (\hat{\alpha}^k, \hat{\beta}^k) \text{ with } (\hat{\alpha}^k, \hat{\beta}^k) \rightarrow (\hat{\alpha}, \hat{\beta}), \quad V(\hat{\alpha}^k, \hat{\beta}^k) \rightarrow V(\hat{\alpha}, \hat{\beta}) \text{ and } (\hat{\alpha}^k, \hat{\beta}^k) \rightarrow (\alpha, \beta) \right\}, \end{aligned} \quad (3.111)$$

and

$$\begin{aligned} {}^\infty \partial^P V(\hat{\alpha}, \hat{\beta}) = & \left\{ (\hat{\lambda}, \hat{\mu}) \in \mathbb{R}^m \times \mathbb{R}^r \mid \exists (\hat{\alpha}^k, \hat{\beta}^k) \wedge \eta_k \downarrow 0 \text{ proximal subgradient of } V(\hat{\alpha}, \hat{\beta}) \right. \\ & \left. \text{at } (\hat{\alpha}^k, \hat{\beta}^k) \text{ with } (\hat{\alpha}^k, \hat{\beta}^k) \rightarrow (\hat{\alpha}, \hat{\beta}), \quad V(\hat{\alpha}^k, \hat{\beta}^k) \rightarrow V(\hat{\alpha}, \hat{\beta}) \text{ and } \eta_k (\hat{\alpha}^k, \hat{\beta}^k) \rightarrow (\alpha, \beta) \right\}, \end{aligned} \quad (3.112)$$

Then $\partial^P V(\hat{\alpha}, \hat{\beta})$ and ${}^\infty \partial^P V(\hat{\alpha}, \hat{\beta})$ are closed sets satisfying $\mathbf{0} \in {}^\infty \partial^P V(\hat{\alpha}, \hat{\beta})$ and it is impossible to have both $\partial^P V(\hat{\alpha}, \hat{\beta}) = \emptyset$ and ${}^\infty \partial^P V(\hat{\alpha}, \hat{\beta}) = \{\mathbf{0}\}$. The formula

$$\partial V(\hat{\alpha}, \hat{\beta}) = \text{cl co} \left[\partial^P V(\hat{\alpha}, \hat{\beta}) + {}^\infty \partial^P V(\hat{\alpha}, \hat{\beta}) \right] \quad (3.113)$$

holds, where

$$\begin{aligned} & \partial^P V(\hat{\alpha}, \hat{\beta}) + {}^\infty \partial^P V(\hat{\alpha}, \hat{\beta}) \\ &= \left\{ (\hat{\alpha}, \hat{\beta}) + (\hat{\alpha}^\circ, \hat{\beta}^\circ) \mid (\hat{\alpha}, \hat{\beta}) \in \partial^P V(\hat{\alpha}, \hat{\beta}), \quad (\hat{\alpha}^\circ, \hat{\beta}^\circ) \in {}^\infty \partial^P V(\hat{\alpha}, \hat{\beta}) \right\}. \end{aligned} \quad (3.114)$$

Further $\partial^P V(\hat{\alpha}, \hat{\beta})$ is the superset of the recession cone to $V(\hat{\alpha}, \hat{\beta})$ at $(\hat{\alpha}, \hat{\beta})$.

The theorem below gives the condition, when $\partial V(\hat{\alpha}, \hat{\beta})$ is bounded and nonempty.

Theorem 3.2.4 [31]

- (a) If every $\mathbf{y} \in \Sigma(\mathbf{0}, \mathbf{0})$ is normal, then V is Lipschitz near $(\mathbf{0}, \mathbf{0})$ and one has

$$\emptyset \neq \partial V(\mathbf{0}, \mathbf{0}) \subset \cup_{\mathbf{y} \in \Sigma(\mathbf{0}, \mathbf{0})} \mathcal{M}^1(\mathbf{y}). \quad (3.115)$$

- (b) If $\mathbf{y}$ is any solution to $P(\mathbf{0}, \mathbf{0})$ then either $\mathbf{y}$ is abnormal or else $\mathcal{M}^1(\mathbf{y}) \neq \emptyset$.

3.2.3 Local Analysis of the Augmented Lagrangian Function

The local properties of the augmented Lagrangian function is analysed by means of its resemblance to the minimisation problem that results from the proximal subdifferential inequality. A vector (λ, μ) is a proximal subgradient to $V(\alpha, \beta)$ if $(\lambda, \mu, -1)$ is actually a proximal normal to the epigraph of $V(\alpha, \beta)$ at $(\lambda, \mu, V(\alpha, \beta))$. It holds if and only if there is function Υ of class $\mathcal{C}^2$ such that $\nabla \Upsilon(\alpha, \beta) = (\lambda, \mu)$ and $V(\alpha, \beta) = \Upsilon(\alpha, \beta)$ in some neighborhood of $(\bar{\alpha}, \bar{\beta})$. The idea of the equivalence of the problems P and P_A goes through showing that $V_A(\alpha, \beta)$ is the function Υ . In the light of the above theorem, if the set $\Sigma_A(\alpha, \beta)$ is a singleton, it is evident that the Lagrange multipliers $\bar{\lambda}$ and $\bar{\mu}$ of $P_A(\alpha, \beta)$ fulfill:

$$\begin{pmatrix} \bar{\lambda}(\alpha) \\ \bar{\mu}(\beta) \\ -\bar{\kappa} \end{pmatrix} \in \mathcal{N}_{\text{epi}(V_A)}(\alpha, \beta) \quad (3.116)$$

in the normal case or

$$\begin{pmatrix} \bar{\lambda}(\alpha) \\ \bar{\mu}(\beta) \\ 0 \end{pmatrix} \in {}^\infty \mathcal{N}_{\text{epi}(V_A)}(\alpha, \beta), \quad (3.117)$$

in the degenerate point. Similarly, the Lagrange multipliers λ and μ of $P(\alpha, \beta)$ fulfill:

$$\begin{pmatrix} \lambda(\alpha) \\ \mu(\beta) \\ -\kappa \end{pmatrix} \in \mathcal{N}_{\text{epi}(V)}(\alpha, \beta) \quad (3.118)$$

in the normal case or

$$\begin{pmatrix} \lambda(\alpha) \\ \mu(\beta) \\ 0 \end{pmatrix} \in {}^\infty \mathcal{N}_{\text{epi}(V)}(\alpha, \beta), \quad (3.119)$$

in the degenerate point.

In order to accomplish this analysis first two important and well-known results of proximal analysis are stated:

Corollary 3.2.2 Let $\mathcal{E} = \text{epi}(V(\alpha, \beta))$ be a nonempty subset of $\mathcal{Y}$, and let $(\lambda, \mu, -\kappa) \in \mathcal{Y}^* \times \mathbb{R}$, $(\alpha, \beta, \nu) \in \mathcal{E}$. The following are equivalent:

1. $(\lambda, \mu, -\kappa) \in \text{proj}_{\mathcal{E}}(\alpha, \beta, \nu)$;
2. $(\lambda, \mu, -\kappa) \in \text{proj}_{\mathcal{E}}((\lambda, \mu, -\kappa) + t((\alpha, \beta, \nu) - (\lambda, \mu, -\kappa))) \forall t \in [0, 1]$;
3. $d_{\mathcal{E}}((\lambda, \mu, -\kappa) + t((\alpha, \beta, \nu) - (\lambda, \mu, -\kappa))) = t \|(\alpha, \beta, \nu) - (\lambda, \mu, -\kappa)\| \forall t \in [0, 1]$;
4. $\langle (\alpha, \beta, \nu) - (\lambda, \mu, -\kappa), (\lambda', \mu', -\kappa') - (\lambda, \mu, -\kappa) \rangle \leq \frac{1}{2} \|(\lambda', \mu', -\kappa') - (\lambda, \mu, -\kappa)\|^2$,
 $\forall (\lambda', \mu', -\kappa') \in \mathcal{E}$

Corollary 3.2.3 Let $\mathcal{E}_{\mathcal{A}} = \text{epi}(V_{\mathcal{A}}(\alpha, \beta))$ be a nonempty subset of $\mathcal{Y}$, and let $(\bar{\lambda}, \bar{\mu}, -\bar{\kappa}) \in \mathcal{Y}^* \times \mathbb{R}$, $(\alpha, \beta, \nu) \in \mathcal{E}_{\mathcal{A}}$. The following are equivalent:

1. $(\bar{\lambda}, \bar{\mu}, -\bar{\kappa}) \in \text{proj}_{\mathcal{E}_{\mathcal{A}}}(\alpha, \beta, \nu)$;
2. $(\bar{\lambda}, \bar{\mu}, -\bar{\kappa}) \in \text{proj}_{\mathcal{E}_{\mathcal{A}}}((\bar{\lambda}, \bar{\mu}, -\bar{\kappa}) + t((\alpha, \beta, \nu) - (\bar{\lambda}, \bar{\mu}, -\bar{\kappa}))) \forall t \in [0, 1]$;
3. $d_{\mathcal{E}_{\mathcal{A}}}((\bar{\lambda}, \bar{\mu}, -\bar{\kappa}) + t((\alpha, \beta, \nu) - (\bar{\lambda}, \bar{\mu}, -\bar{\kappa}))) = t \|(\alpha, \beta, \nu) - (\bar{\lambda}, \bar{\mu}, -\bar{\kappa})\| \forall t \in [0, 1]$;
4. $\langle (\alpha, \beta, \nu) - (\bar{\lambda}, \bar{\mu}, -\bar{\kappa}), (\bar{\lambda}', \bar{\mu}', -\bar{\kappa}') - (\bar{\lambda}, \bar{\mu}, -\bar{\kappa}) \rangle \leq \frac{1}{2} \|(\bar{\lambda}', \bar{\mu}', -\bar{\kappa}') - (\bar{\lambda}, \bar{\mu}, -\bar{\kappa})\|^2$
 $\forall (\bar{\lambda}', \bar{\mu}', -\bar{\kappa}') \in \mathcal{E}_{\mathcal{A}}$

The value function $V_{\mathcal{A}}(\alpha, \beta)$ is C^1 in its arguments and thereby the Gateaux derivative exists. Further if the set and $\Phi_{\mathcal{A}}(\alpha, \beta) \neq \emptyset$ then $V_{\mathcal{A}}(\alpha, \beta) < +\infty$ and if at (α, β) $V_{\mathcal{A}}(\alpha, \beta)$ is calm, then $V_{\mathcal{A}}(\alpha, \beta) > -\infty$. The evaluation of Clarke's generalised directional derivative reveals:

$$V_{\mathcal{A}}^o(\alpha, \beta; \hat{\alpha}_i) = \bar{\lambda}_i(\alpha) + c(\mathbf{h}_i(\mathbf{y}) + \alpha_i), \quad \text{for } i = 1, \dots, m, \quad (3.120)$$

$$V_{\mathcal{A}}^o(\alpha, \beta; \hat{\beta}_j) = \bar{\mu}_j(\beta) + c(\mathbf{g}_j(\mathbf{y}) + \beta_j), \quad \text{for } j = 1, \dots, n. \quad (3.121)$$

Consideration of the inclusions in (3.118) and (3.119) together with the Gateaux derivatives of the value function as given in equations (3.120) to (3.121) together reveals:

$$\bar{\lambda}_i(\alpha) = \bar{\lambda}_i(\alpha) + c(\mathbf{h}_i(\mathbf{y}) + \alpha_i), \quad \text{for } i = 1, \dots, m, \quad (3.122)$$

$$\bar{\mu}_j(\beta) = \bar{\mu}_j(\beta) + c(\mathbf{g}_j(\mathbf{y}) + \beta_j), \quad \text{for } j = 1, \dots, n. \quad (3.123)$$

The formulas given in (3.122) and (3.123) provide external estimates for the dual multipliers in intermediate steps of minimisation of the augmented Lagrangian. Another important feature of the equalities in (3.122) and (3.123) is that they are valid irrespective whether the problem P is normal or abnormal. The proximal subgradient inequality implies following:

$$V(\boldsymbol{\alpha}, \boldsymbol{\beta}) - V(\mathbf{0}, \mathbf{0}) + \sigma \|\boldsymbol{\alpha}\|^2 + \sigma \|\boldsymbol{\beta}\|^2 \geq \langle \boldsymbol{\lambda}, \boldsymbol{\alpha} \rangle + \langle \boldsymbol{\mu}, \boldsymbol{\beta} \rangle. \quad (3.124)$$

The point $\mathbf{y}_0 \in \Sigma(\mathbf{0}, \mathbf{0})$ satisfies $f(\mathbf{y}_0) = V(\mathbf{0}, \mathbf{0})$. If $\mathbf{y}$ is sufficiently close to $\mathbf{y}_0$ by the Lipschitz continuity of $\mathbf{h}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$, approximations $\mathbf{h}(\mathbf{y}) \approx -\boldsymbol{\alpha}$ and $\mathbf{g}(\mathbf{y}) \approx -\boldsymbol{\beta}$ are valid. So the proximal subgradient inequality takes the after rearranging the following form as given in [31]:

$$f(\mathbf{y}) + \langle \boldsymbol{\lambda}, \mathbf{h}(\mathbf{y}) \rangle + \langle \boldsymbol{\mu}, \mathbf{g}(\mathbf{y}) \rangle + \sigma \|\mathbf{h}(\mathbf{y})\|^2 + \sigma \|\mathbf{g}(\mathbf{y})\|^2 \geq f(\mathbf{y}_0) \quad (3.125)$$

This is an inequality which holds for all $\mathbf{y}$ in the vicinity of $\mathbf{y}_0$. This is equivalent to say that the following function attains a local minimum at $\mathbf{y}_0$ for some σ :

$$\mathbf{y} \longmapsto f(\mathbf{y}) + \langle \boldsymbol{\lambda}, \mathbf{h}(\mathbf{y}) \rangle + \langle \boldsymbol{\mu}, \mathbf{g}(\mathbf{y}) \rangle + \sigma \|\mathbf{h}(\mathbf{y})\|^2 + \sigma \|\mathbf{g}(\mathbf{y})\|^2. \quad (3.126)$$

which is equivalent to:

$$\mathbf{y} \longmapsto f(\mathbf{y}) + \langle \boldsymbol{\lambda}, \mathbf{h}(\mathbf{y}) \rangle + \langle \boldsymbol{\mu}, \mathbf{g}(\mathbf{y}) \rangle + \sigma \sum_{i=1}^m \mathbf{h}_i(\mathbf{y})^2 + \sigma \sum_{j=1}^r \max(0, \mathbf{l}_j(\mathbf{y}))^2. \quad (3.127)$$

If σ is taken as the set of values that are greater or equal than the threshold value of penalty factor c , for which P_A becomes equivalent to P , then this form is equivalent to the representation of the augmented Lagrangian function:

$$\mathbf{y} \longmapsto f(\mathbf{y}) + \langle \boldsymbol{\lambda}, \mathbf{h}(\mathbf{y}) \rangle + \langle \boldsymbol{\mu}, \mathbf{g}(\mathbf{y}) \rangle + c \sum_{i=1}^m \mathbf{h}_i(\mathbf{y})^2 + c \sum_{j=1}^r \max(0, \mathbf{l}_j(\mathbf{y}))^2. \quad (3.128)$$

As a corollary, one sees that the local minimum of problem P coincides with the minimum of P_A for a $\epsilon > 0$ such that $\mathbf{y} \in \mathbf{y}_0 + \mathbb{B}(\mathbf{y}_0, \epsilon)$. Further, by the properties of the proximal subdifferential inequality it is seen that the dual multipliers of $P_A(\mathbf{0}, \mathbf{0})$ at $\mathbf{y}_0$ coincide with the dual multipliers of $P(\mathbf{0}, \mathbf{0})$. From which one can deduce that :

$$\lim_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \rightarrow (\mathbf{0}, \mathbf{0})} V_A(\boldsymbol{\alpha}, \boldsymbol{\beta}) = V(\mathbf{0}, \mathbf{0}) = \min_{\mathbf{h}=\mathbf{0}, \mathbf{g}=\mathbf{0}} f(\mathbf{y}^*) \quad (3.129)$$

If $\mathbf{y}^* \in \Sigma(\mathbf{0}, \mathbf{0})$ is a minimum at which the minimum is regular then:

$$\begin{pmatrix} \bar{\boldsymbol{\lambda}}(\mathbf{0}) \\ \bar{\boldsymbol{\mu}}(\mathbf{0}) \\ -1 \end{pmatrix} \in \mathcal{N}_{\text{epi}(V)}(\mathbf{0}, \mathbf{0}). \quad (3.130)$$

or if $\mathbf{y}^* \in \Sigma(\mathbf{0}, \mathbf{0})$ is a minimum at which the minimum is degenerate then, one has:

$$\begin{pmatrix} \bar{\lambda}(\mathbf{0}) \\ \bar{\mu}(\mathbf{0}) \\ 0 \end{pmatrix} \in \mathcal{N}_{\text{epi}(V)}^\infty(\mathbf{0}, \mathbf{0}). \quad (3.131)$$

At $\mathbf{y}_0$ following relation is then valid:

$$\lambda(\mathbf{0}) = \bar{\lambda}(\mathbf{0}) + c \mathbf{h}(\mathbf{y}_0), \quad (3.132)$$

$$\mu(\mathbf{0}) = \bar{\mu}(\mathbf{0}) + c \mathbf{g}(\mathbf{y}_0). \quad (3.133)$$

3.2.4 Global Analysis of the Augmented Lagrangian Function

The Lagrangian has a convex-concave saddle structure. If one tries to minimise the Lagrangian with respect to the primal variable vector $\mathbf{y}$ starting from an arbitrary point, in case that the search directions tend into concave regions, the search may converge to $-\infty$. A remedy to this problem is provided by the augmented Lagrangian formulation if the constraints fulfill several conditions. The penalisation in the augmented Lagrangian function induces a local convexification around optimal primal and dual vectors as discussed in [14]. The quadratic penalty terms in the goal function cause augmented Lagrangian algorithms to "slide" along feasible sets. In the sequel this local convexification is generalised to the Lipschitzian nonsmooth setting.

If the feasible region described by the constraints of the NLP is closed and bounded, then this induces also for sufficiently high penalty factors a convexification of the feasible region by the additional terms in the goal function in (3.82) under some assumptions. By the convexification of the feasible region, the divergence to $-\infty$ is prevented. The global behaviour of the augmented Lagrangian schemes can best be understood by first considering a more simplified case where the nonlinear programming problem stated in (3.73) to (3.75) is smooth and has only twice continuously differentiable equality constraints. This simplified smooth analysis is also found in [14]. Then the augmented Lagrangian function in (3.82) becomes:

$$\mathcal{L}_a = \kappa f(\mathbf{y}) + \langle \bar{\lambda}, \mathbf{h}(\mathbf{y}) \rangle + \frac{1}{2} c \sum_{i=0}^m h_i^2(\mathbf{y}). \quad (3.134)$$

The gradient and the Hessian of (3.134) are given by:

$$\nabla_{\mathbf{y}} \mathcal{L}_a(\mathbf{y}, \boldsymbol{\lambda}) = \kappa \nabla_{\mathbf{y}} f(\mathbf{y}) + \nabla_{\mathbf{y}} \mathbf{h}(\mathbf{y}) (\boldsymbol{\lambda} + c \mathbf{h}(\mathbf{y})) \quad (3.135)$$

and

$$\nabla_{\mathbf{y}\mathbf{y}}^2 \mathcal{L}_a(\mathbf{y}, \boldsymbol{\lambda}) = \kappa \nabla_{\mathbf{y}\mathbf{y}}^2 f(\mathbf{y}) + \sum_{i=1}^m (\boldsymbol{\lambda}_i + c \mathbf{h}_i(\mathbf{y})) \nabla_{\mathbf{y}\mathbf{y}}^2 \mathbf{h}_i(\mathbf{y}) + c \nabla_{\mathbf{y}} \mathbf{h} \nabla_{\mathbf{y}} \mathbf{h}^T, \quad (3.136)$$

respectively. The second order sufficiency conditions for this smooth problem take the following form:

Theorem 3.2.5: Second Order Sufficiency Conditions in the Smooth Case [14]

Assume that f and $\mathbf{h}$ are twice continuously differentiable, and let $\mathbf{y}^* \in \mathbb{R}^n$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ satisfy

$$\nabla_{\mathbf{y}}\mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*) = 0, \quad \nabla_{\boldsymbol{\lambda}}\mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*) = 0, \quad (3.137)$$

$$\mathbf{z}^T \nabla_{\mathbf{y}\mathbf{y}}^2 \mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*) \mathbf{z} > 0, \quad \forall \mathbf{z} \neq \mathbf{0} \quad \text{with} \quad \nabla_{\mathbf{y}} \mathbf{h}(\mathbf{y})^T \mathbf{z} = \mathbf{0}. \quad (3.138)$$

Then $\mathbf{y}^*$ is a strict local minimum of f subject to $\mathbf{h}(\mathbf{y}) = \mathbf{0}$. In fact there exists scalars $\gamma > 0$ and $\epsilon > 0$ such that

$$f(\mathbf{y}) \geq f(\mathbf{y}^*) + \frac{\gamma}{2} \|\mathbf{y} - \mathbf{y}^*\|^2, \quad \forall \mathbf{y} \quad \text{with} \quad \mathbf{h}(\mathbf{y}) = \mathbf{0} \quad \text{and} \quad \|\mathbf{y} - \mathbf{y}^*\| < \epsilon. \quad (3.139)$$

In order to obtain a generalisation of theorem to the nonsmooth Lipschitzian setting several assumptions are needed.

Hypothesis 3.2.2

Let $\mathbf{y}^*$ be an element of $\Sigma_{\mathcal{A}}(\mathbf{0}, \mathbf{0})$ and let for all $\mathbf{y}_i \in \mathbb{B}(\mathbf{y}^*, \epsilon)$ be valid:

- Each $\nabla f(\mathbf{y}_i)$ is Lipschitzian with lipschitz coefficients $[L_f^1(\mathbf{y}_i), \dots, L_f^n(\mathbf{y}_i)]$, $\forall \mathbf{y} + \epsilon \mathbb{B}$.
- Each $J h(\mathbf{y}_i)$ is Lipschitzian $\forall \mathbf{y} + \epsilon \mathbb{B}$ with lipschitz coefficients $[L_h^1(\mathbf{y}_i), \dots, L_h^n(\mathbf{y}_i)]$.
- Each $J g(\mathbf{y}_i)$ is Lipschitzian $\forall \mathbf{y} + \epsilon \mathbb{B}$ with lipschitz coefficients $[L_g^1(\mathbf{y}_i), \dots, L_g^n(\mathbf{y}_i)]$.

If hypothesis 1 is valid, then the generalised Hessian $\partial_{\mathbf{y}\mathbf{y}}\mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ is compact. The generalised Hessian of $\partial_{\mathbf{y}\mathbf{y}}\mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ is defined as follows:

$$\partial_{\mathbf{y}\mathbf{y}}\mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*) = \left\{ \Pi(\mathbf{y}, \boldsymbol{\lambda}; \mathbf{z}, \mathbf{w}) \mid \mathcal{L}^{00}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \mathbf{v}, \mathbf{w}) \geq \langle \mathbf{z}, \Pi(\mathbf{y}, \boldsymbol{\lambda}; \mathbf{z}, \mathbf{w}) \mathbf{w} \rangle \quad \forall \mathbf{z}, \forall \mathbf{w} \right\}. \quad (3.140)$$

Here the linear operator $\Pi(\mathbf{y}, \boldsymbol{\lambda}; \mathbf{z}, \mathbf{w})$ is an element of the generalised Hessian $\partial_{\mathbf{y}\mathbf{y}}\mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ of the Lagrangian. The compactness of $\partial_{\mathbf{y}\mathbf{y}}\mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ imply that there exist constants $\mathbf{K} = [K_1, \dots, K_n]$ such that

$$\partial_{\mathbf{y}\mathbf{y}}\mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \subset \mathbf{K} \mathbb{B}_{n \times n} \quad (3.141)$$

The second-order Clarke's generalised derivative of a Lagrangian of the form (3.76) is defined as:

$$\mathcal{L}^{00}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \mathbf{v}, \mathbf{w}) = \limsup_{\substack{\mathbf{y}_k \rightarrow \mathbf{y} \\ t \downarrow 0, s \downarrow 0 \\ \mathbf{v}_i \rightarrow \mathbf{v}, \quad \mathbf{w}_j \rightarrow \mathbf{w}}} \frac{\mathcal{L}(\mathbf{y}_k + t \mathbf{v}_i + s \mathbf{w}_j, \boldsymbol{\lambda}, \boldsymbol{\mu}) - \mathcal{L}(\mathbf{y} + t \mathbf{v}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{s t}. \quad (3.142)$$

The limiting subdifferential and the limiting Hessian of (3.134) in the Lipschitzian setting are given by:

$$\partial_{\mathbf{y}} \mathcal{L}_a(\mathbf{y}, \boldsymbol{\lambda}) \subset \text{co cl} (\kappa \partial_{\mathbf{y}} f(\mathbf{y}) + \partial_{\mathbf{y}} \mathbf{h}(\mathbf{y}) (\boldsymbol{\lambda} + c \mathbf{h}(\mathbf{y}))) \quad (3.143)$$

and

$$\partial_{\mathbf{y}\mathbf{y}}^2 \mathcal{L}_a(\mathbf{y}, \boldsymbol{\lambda}) \subset \text{co cl} \left(\kappa \partial_{\mathbf{y}\mathbf{y}}^2 f(\mathbf{y}) + \sum_{i=1}^m (\boldsymbol{\lambda}_i + c \mathbf{h}_i(\mathbf{y})) \partial_{\mathbf{y}\mathbf{y}}^2 \mathbf{h}_i(\mathbf{y}) + c \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \right), \quad (3.144)$$

respectively. Here $\boldsymbol{\xi}$ an element of is an element of $\partial_{\mathbf{y}} \mathbf{h}(\mathbf{y})$ and the set $\mathcal{S}_{\mathbf{h}}$ is defined by:

$$\mathcal{S}_{\mathbf{h}} = \{ \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \mid \boldsymbol{\xi} \in \partial_{\mathbf{y}} \mathbf{h}(\mathbf{y}) \}, \quad (3.145)$$

Analogously, the the limiting subdifferential and the limiting Hessian of of the Lagrangian in the Lipschitzian setting are given by:

$$\partial_{\mathbf{y}} \mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}) \subset \text{co cl} (\kappa \partial_{\mathbf{y}} f(\mathbf{y}) + \boldsymbol{\lambda} \partial_{\mathbf{y}} \mathbf{h}(\mathbf{y})) \quad (3.146)$$

and

$$\partial_{\mathbf{y}\mathbf{y}}^2 \mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}) \subset \text{co cl} \left(\kappa \partial_{\mathbf{y}\mathbf{y}}^2 f(\mathbf{y}) + \sum_{i=1}^m \boldsymbol{\lambda}_i \partial_{\mathbf{y}\mathbf{y}}^2 \mathbf{h}_i(\mathbf{y}) \right), \quad (3.147)$$

respectively. Since for sufficiently high penalty parameters the cost of infeasibility dominates the goal function, as the optimisation proceeds one has:

$$\lim_{\substack{\|\mathbf{h}(\mathbf{y})\| \rightarrow 0 \\ \|\mathbf{g}(\mathbf{y})\| \rightarrow 0}} \partial_{\mathbf{y}} \mathcal{L}_{\mathcal{A}} \longrightarrow \partial_{\mathbf{y}} \mathcal{L}, \quad (3.148)$$

and

$$\lim_{\substack{\|\mathbf{h}(\mathbf{y})\| \rightarrow 0 \\ \|\mathbf{g}(\mathbf{y})\| \rightarrow 0}} \partial_{\mathbf{y}\mathbf{y}} \mathcal{L}_{\mathcal{A}} \subset \text{co cl} (\partial_{\mathbf{y}\mathbf{y}} \mathcal{L} + c \mathcal{S}_{\mathbf{h}} + c \mathcal{S}_{\mathbf{g}}), \quad (3.149)$$

The elements of $\partial_{\mathbf{y}\mathbf{y}} \mathcal{L}$ are second-order tensors with bounded spectrum under the validity of the hypothesis 3.2.2. The elements of $\mathcal{S}_{\mathbf{h}}$ are positive semi-definite tensors of order two.

Let $\mathbf{y}^* \in \Sigma(0)$ and $\boldsymbol{\lambda}^* \in \mathcal{M}^1(\mathbf{y}^*)$. If for all $\mathbf{v}$ for which:

$$\langle \mathbf{v}, \Theta \mathbf{v} \rangle = 0, \quad \forall \Theta \in \mathcal{S}_{\mathbf{h}}(\mathbf{y}^*). \quad (3.150)$$

the condition

$$\langle \mathbf{v}, \Theta \mathbf{v} \rangle > 0, \quad \forall \Theta \in \partial_{\mathbf{y}\mathbf{y}} \mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*) \quad (3.151)$$

is valid. Then by invoking Lemma 3.2.6, there exists for every $\Phi \in \partial_{\mathbf{y}\mathbf{y}} \mathcal{L}(\mathbf{y}^*, \boldsymbol{\lambda}^*)$, a c and a positive-semidefinite $\Theta \in \mathcal{S}_{\mathbf{h}}$ such that all elements of $\boldsymbol{\Psi} \in \partial_{\mathbf{y}\mathbf{y}} \mathcal{L}_{\mathcal{A}}(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ become positive definite:

$$\langle \mathbf{w}, \boldsymbol{\Psi} \mathbf{w} \rangle > 0, \quad \forall \boldsymbol{\Psi} \in \partial_{\mathbf{y}\mathbf{y}} \mathcal{L}_{\mathcal{A}}(\mathbf{y}, \boldsymbol{\lambda}), \quad \forall \mathbf{w} \neq \mathbf{0}. \quad (3.152)$$

This gives the condition under which a local minimum of problem P , is convexified nonsmoothly if it is characterised in terms of its augmented Lagrangian function. This result gives the conditions when a nonconvex local minimum is convexified when the search directions are projected to the tangent space $\mathcal{T}_{\mathbf{h}=\mathbf{0}}(\mathbf{y})$ for some $\epsilon > 0$ and $\|\mathbf{y}^* - \mathbf{y}\| \in \mathbb{B}(\mathbf{y}^*, \epsilon)$.

3.2.5 Comparison of Strategies for the Solution of the Unconstrained Minimisation

There are several aspects that make the augmented Lagrangian approach favorable in comparison to other approaches. In exact penalty methods, instead of performing a sequence of minimisations, a single minimisation is performed but the penalty parameter has to be set very high, such that ill-conditioning is caused, and depending on the minimisation method used the condition number of the Hessian matrix is severed. The sequential minimisation besides preventing ill-conditioning, allows partial minimisations especially at the initial stages of the successive minimisations such that the successive minimisations proceed faster than expected. The global optimizing property as in the exact penalty approach is however preserved. The augmented Lagrangian is a method that is numerically compliant with the underlying penalty approach. The Newton method in the minimisation of unconstrained functions is a favored method because of its superlinear convergence in the vicinity of the solution. However, in large scale optimisation problems where the structure of the Hessian matrix is not sparse, the evaluation of the Hessian matrix becomes cumbersome. In the case of the augmented Lagrangian technique, in general, by the convexification induced by adding quadratic penalty terms to the goal function, the structure of the Hessian matrix becomes dense. In such cases, quasi-Newton methods are used in order to extrapolate the Hessian matrix by a formula like BFGS method. If the function to be minimised is nonsmooth, then the nonuniqueness and unboundedness of the Hessian may pose severe problems. A way to circumvent the problem of the nonuniqueness and unboundedness of the Hessian matrix induced by using reformulation functions is to use smoothing. This has the disadvantage however that the smoothing parameter has to be adjusted in the course of the optimisation, which reduces the speed of the algorithm by introducing additional loops in which the smoothing parameter is adjusted. The modified conjugate gradient methods offer a trade-off by performing a minimisation based on second-order information without calculating them. The only required information is first-order information such as the gradient and second-order information such that estimates of the Hessian are extrapolated numerically, and it is not necessary to calculate the dense Hessian matrix. The application of the augmented Lagrangian method to nonlinear programming problems with equilibrium constraints is not wide spread. But the linear complementarity equality condition is being linearised and relaxed first which introduces another burden except limiting the upper value of the penalty factor. Below the main aspects in the comparison of different optimisation

approaches is presented:

1. One of the main advantages of direct-search methods is that the minimisation is performed by function value comparison, so that the problems of ill-conditioning due to high penalty parameters that degrade the performance of methods that use first and second-order information, is not a problem for such methods.
2. The choice of a high penalty value has generally a convexifying effect so if the original problem is not convex iterative first-order methods could face the dilemma of destabilizing due to high c and divergence of low c . In this respect direct methods are more suitable for a general class of optimisation problems. Direct search methods have the disadvantage that their convergence in the vicinity of the solution is slow. A partial remedy to this problem in the framework of the augmented Lagrangian technique is the partial/incomplete minimisation approach in the intermediate stages of the successive minimisations.
3. The local convexification of the feasible region, that is induced by the penalisation, guarantees that the unconstrained function is bounded-below and that through direct-search methods the some feasible region is reached. Moreover, first-order methods can get stuck in locally minimal solutions depending on severity of the nonconvexity of the value function and direct-search methods are less prone to such a problem.
4. Newton methods are preferred because of their fast convergence in the vicinity of the solution. However, they are favorable in case of sparsity of the structure of the Hessian. In general, for augmented Lagrangian functions the Hessian matrix is highly dense, which impairs execution times for higher dimensional nonlinear programming problems .
5. Another issue for the Newton methods is the boundedness of the norm of the Hessian matrix at the solution. Some reformulation functions, such as the Fischer-Burmeister function, do not have bounded Hessians, and smoothing is necessary to implement those together with Newton methods.
6. Sequential Quadratic Programming (SQP) relies on exploiting the resemblance of a smooth nonlinear programming problem, to a standard Quadratic Programming (QP) problem. The resemblance is achieved by successive linearisations of goal functions and constraints. For nonsmooth functions these linearisations may provide less accurate approximations at solution points.
7. In exact penalty methods, instead of performing a sequence of minimisations, a single minimisation is performed but the penalty parameter has to be set very high, such that

ill-conditioning is caused, and depending on the minimisation method used the condition number of the Hessian matrix is severed.

8. Bundle methods are commonly featuring in nonsmooth optimisation. They require intensive information storage and processing of the subgradients, thus their efficiency and computational speed is impaired by optimisation problems with more than 500 variables.

3.3 A NLP Scheme For The Calculation of Discontinuous Optimal Trajectories for FDLS with Blockable Degrees of Freedom

The sweeping discretisation method has been since its introduction in the late sixties been investigated intensively as an integration routine. The aim is to investigate the properties of the sweeping discretisation scheme from the point of optimisation, specifically from the point of trajectory optimisation. The integration routines that are developed based on the sweeping discretisation scheme are examined on the accuracy of the integration in literature. A particular interest is to determine switching times very accurately, at events such as opening of a contact or stick-slip transitions at a contact. This is to exploit the quantitative properties of the sweeping integration method as much as possible, but in the aspect of accuracy the event-driven approaches excel the sweeping discretisation schemes. The advantage of using sweeping discretisation in the trajectory optimisation problem is the qualitative advantages it offers with respect to other optimisation approaches. By treating the contact forces and the forces that induce transition as impulsive control and contact forces the distributional sense, one obtains in some cases the mode sequence and optimal transition instants and locations in advance without prespecifying them. The existence of switching control forces in the distributional sense, implies that the transition indeed happen on a time-instant of Lebesgue measure zero if one considers the continuous description of the problem, on the other side in the discretisation it means that the transition happens on a discrete time interval on which the discretised differential measures of control forces exist in the distributional sense. The continuous description is compliant to various hybrid system modeling approaches where transitions happen on a time instant of Lebesgue-measure zero, whereas the discretised treatment the problem enables the transition to take place on a time interval of length of the discretisation length. In literature there several streams that treat in detail transitions on intervals with nonzero Lebesgue measure in order to stay physical reality and deny conventional hybrid system descriptions by making use of examples from the physical world. The measure-differential inclusion approach and the sweeping discretisation comply in the continuous description with convential hybrid system modeling approaches which are reasonable, whereas the discretised treatment accepts transitions on in-

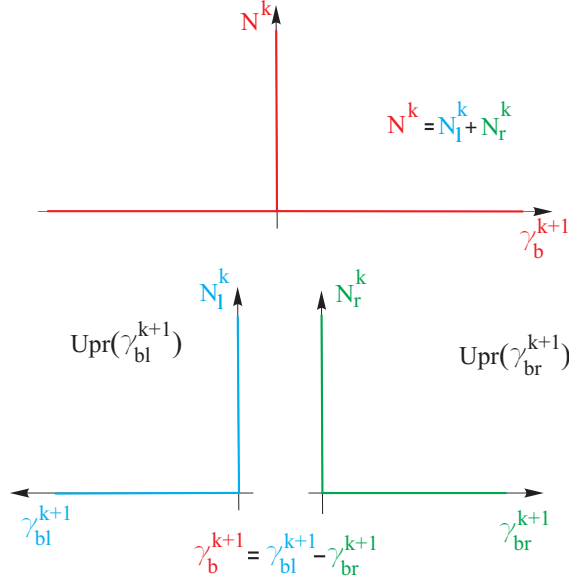


Figure 3.4: The decomposition of the Upr relation into two unilateral primitives in discretised form.

tervals. So by choosing minimum or maximum of discretisation step length it is possible to limit transitions such that they comply with the physical system which is investigated. In the opinion of the author, therefore it makes less sense to use sweeping discretisation if the aim is to achieve numerical accuracy in the contact state transitions because of the existence of contact forces in the distributional sense, and it recommends itself to revert to event-driven integration schemes, whereas in order to obtain basically a robust treatment of a fairly complex dynamical process sweeping discretisation scheme is superior in providing quantitative data on the problem that compensates for the loss in the numerical accuracy. The high-level of numerical accuracy is on the other hand unrealistic from the physical point of view most of the time. One such superior advantage of sweeping integration techniques is the to provide statistically accurate, experimentally verifiable information on granular flow consisting of many particles, where the time evolution of singular particle is not relevant from the global view of the system. By this approach modeling physical reality, the discretisation step length and its upper and lower boundary has a physical significance. Choosing a discretisation step length which is much smaller then any fast process that takes place in the real system under consideration does not provide further information and may complicate the numerical execution.

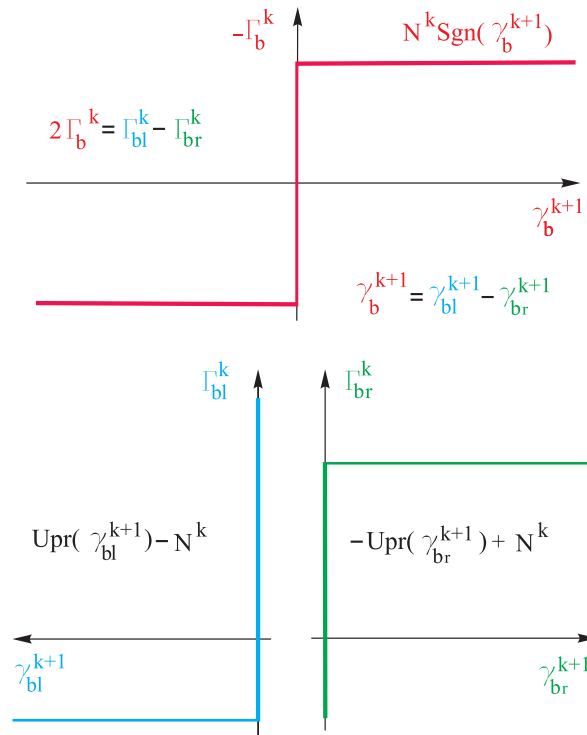


Figure 3.5: The set-valued signum relation and its decomposition into two unilateral primitives that represents relation between the discretised differential measure of normal blocking force to the discretised differential measure of the control force in discretised form.

3.3.1 The Transcription of a mechanical MDI Optimal Control Problem into a NLP

The problem is discretised in time over N equidistant points. The discretised form of the measure-differential inclusion representation is stated in relations (2.120) to (2.123) are given in relations (3.153) to (3.156):

$$\mathbf{M}(\mathbf{q}_m^k) (\mathbf{u}^{k+1} - \mathbf{u}^k) - \mathbf{h}(\mathbf{q}_m^k, \mathbf{u}^k) \Delta t - \mathbf{B}(\mathbf{q}_m^k) \boldsymbol{\Gamma}^k = \mathbf{0}, \quad k = 1, \dots, N, \quad (3.153)$$

$$-\Gamma_{b_i}^k \in N_i^k \text{Sgn}(\gamma_{b_i}^{k+1}), \quad \forall i \in \mathcal{I}_B, \quad k = 1, \dots, N, \quad (3.154)$$

$$|\gamma_{b_i}^{k+1}| \in \text{Up}r(N_i^k), \quad \forall i \in \mathcal{I}_B, \quad k = 1, \dots, N, \quad (3.155)$$

$$\boldsymbol{\tau}^k \in \mathcal{C}_\tau, \quad k = 1, \dots, N. \quad (3.156)$$

The discretised differential measures of $d\Gamma_{b_{r_i}}, d\Gamma_{b_{l_i}}, dN_i, d\Gamma_{b_i}$ are given by $\boldsymbol{\Gamma}_{b_{r_i}}, \boldsymbol{\Gamma}_{b_{l_i}}, \mathbf{N}_i, \boldsymbol{\Gamma}_{b_i}$. These discretised differential measures exist in the distributional sense on a given discretisation time interval. The position dependent entities in the discretised measure-differential inclusion of the controlled mechanical system such as mass matrix $\mathbf{M}$, vector $\mathbf{h}$, $\mathbf{B}$ are evaluated at $\mathbf{q}_m$. The force vector $\mathbf{h}$ is further evaluated at $\mathbf{u}^k$. The midpoint $\mathbf{q}_m^k$ is given by:

$$\mathbf{q}_m^k = \mathbf{q}^k + \frac{\Delta t}{2} \mathbf{u}^k, \quad (3.157)$$

and the discretisation in a time interval is given by:

$$\mathbf{q}_m^{k+1} = \mathbf{q}_m^k + \frac{\Delta t}{2} (\mathbf{u}^{k+1} + \mathbf{u}^k). \quad (3.158)$$

The representation of the set-valued control law stated in equations (3.154) and (3.155) in discretised form is stated in relations (3.159) to (3.165) as a linear complementarity in the given variables:

$$\Gamma_{b_{r_i}}^k = \Gamma_{b_i}^k + N_i^k, \quad k = 1, \dots, N, \quad \forall i \in \mathcal{I}_B, \quad (3.159)$$

$$\Gamma_{b_{l_i}}^k = -\Gamma_{b_i}^k + N_i^k, \quad k = 1, \dots, N, \quad \forall i \in \mathcal{I}_B, \quad (3.160)$$

$$\gamma_{b_i}^{k+1} = \gamma_{b_{r_i}}^{k+1} - \gamma_{b_{l_i}}^{k+1}, \quad k = 1, \dots, N, \quad \forall i \in \mathcal{I}_B, \quad (3.161)$$

$$\gamma_{b_{r_i}}^{k+1} \Gamma_{b_{r_i}}^k = 0, \quad \gamma_{b_{r_i}}^{k+1} \geq 0, \quad \Gamma_{b_{r_i}}^k \geq 0, \quad k = 1, \dots, N, \quad \forall i \in \mathcal{I}_B, \quad (3.162)$$

$$\gamma_{b_{l_i}}^{k+1} \Gamma_{b_{l_i}}^k = 0, \quad \gamma_{b_{l_i}}^{k+1} \geq 0, \quad \Gamma_{b_{l_i}}^k \geq 0, \quad k = 1, \dots, N, \quad \forall i \in \mathcal{I}_B, \quad (3.163)$$

$$\gamma_{b_{r_i}}^{k+1} N_i^k = 0, \quad \gamma_{b_{r_i}}^{k+1} \geq 0, \quad N_i^k \geq 0, \quad k = 1, \dots, N, \quad \forall i \in \mathcal{I}_B, \quad (3.164)$$

$$\gamma_{b_{l_i}}^{k+1} N_i^k = 0, \quad \gamma_{b_{l_i}}^{k+1} \geq 0, \quad N_i^k \geq 0, \quad k = 1, \dots, N, \quad \forall i \in \mathcal{I}_B. \quad (3.165)$$

The set valued relations which are stated in (3.154) and (3.155) are resolved by means of the affine relations stated in equations (3.159) to (3.161) and by the mutual exclusivity conditions stated in equations (3.162) to (3.165). The discretised forms of set-valued control laws depicted

in figures 2.10 and 2.6 can be seen in figures 3.4 and 3.5, respectively. The above discretisation scheme is used for every DOF of the generalised coordinates. In the preparation of the numerical method, the aim is to reduce the number of variables and constraints in order to decrease execution time and inaccuracies in the numerical results. By making use of equalities given in equations from (3.159) to (3.161) the variables $\Gamma_{br_i}^k$, $\Gamma_{bl_i}^k$ and $\gamma_{bl_i}^{k+1}$ are eliminated. Further, by making use of the affine relation between generalised position and velocities as given in equation (3.158) the generalised positions are eliminated from the set of variables. More advanced discretisation schemes may be found in the literature, such as the powerful Θ -method, an algorithm based on displacements with proven convergence. The literature on the sweeping based simulation of systems with friction and impacts is vast and a good overview is provided in [7], [97], [98], [99], [100]. In order to achieve better efficiency from the optimisation algorithm several steps are performed:

- By making use of the equalities stated in equations (3.159) to (3.161) variables are eliminated.
- All equality constraints that occur in the mathematical programming problem are transformed into inequalities.
- By making use of the discretisation scheme in time, the generalised positions are expressed in terms of generalised velocities and are eliminated as variables.
- The complementarities in (3.159) to (3.165) after preparation and elimination of some variables are reformulated by the Fischer-Burmeister function.

For a scleronomic mechanical system with p blockable degrees of freedom, s -dimensional Lebesgue measurable controls and f degrees of freedom, the nonlinear programming problem is set up.

The optimal control problem in finite dimensional form is stated as below:

$$\min_{\mathbf{y}} f(\mathbf{y}) \quad k = 1, \dots, N, \quad (3.166)$$

$$\begin{aligned} B^{k,j} : & \sum_{l=1}^f \mathbf{m}^{k,l}(\mathbf{q}_m^k, \omega)(\mathbf{u}^{k+1,l} - \mathbf{u}^{k,l}) - \mathbf{h}(\mathbf{q}_m^k, \omega, \mathbf{u}^k) \Delta t \\ & - \sum_{l=1}^p \mathbf{w}_l^j(\mathbf{q}_m^k, \omega) \Gamma_l^k - \sum_{l=1}^s \mathbf{b}_l^j(\mathbf{q}_m^k, \omega) \tau_l^k = 0, \quad j = 1, \dots, f, \end{aligned} \quad (3.167)$$

$${}_R\Phi^{k,j} : \Phi(\gamma_{br}^{k+1,j}, \Gamma^{k,j} + \mathbf{N}^{k,j}) = 0, \quad j = 1, \dots, p, \quad (3.168)$$

$${}_L\Phi^{k,j} : \Phi(\gamma_{br}^{k+1,j} - (\mathbf{w}^j(\mathbf{q}_m^k, \omega))^T \mathbf{u}^{k+1}, \mathbf{N}^{k,j} - \Gamma^{k,j}) = 0, \quad j = 1, \dots, p, \quad (3.169)$$

$${}_{NR}\Phi^{k,j} : \Phi(\gamma_{br}^{k+1,j}, \mathbf{N}^{k,j}) = 0, \quad j = 1, \dots, p, \quad (3.170)$$

$${}_{NL}\Phi^{k,j} : \Phi(\gamma_{br}^{k+1,j} - (\mathbf{w}^j(\mathbf{q}_m^k, \omega))^T \mathbf{u}^{k+1}, \mathbf{N}^{k,j}) = 0, \quad j = 1, \dots, p, \quad (3.171)$$

$$\tau_{\min}^{k,j} \leq \tau^{k,j} \leq \tau_{\max}^{k,j}, \quad j = 1, \dots, s, \quad (3.172)$$

$$V_j : \mathbf{q}_j^{N+1} - \mathbf{q}_d^j = 0, \quad j = 1, \dots, f, \quad (3.173)$$

$$V_{j+f} : \mathbf{u}_j^{N+1} - \mathbf{u}_d^j = 0, \quad j = 1, \dots, f, \quad (3.174)$$

$$t_{\min} \leq t_f \leq t_{\max}. \quad (3.175)$$

Here $\mathbf{m}^{k,l}(\mathbf{q}_m^k, \omega)$ denotes the l th of the mass matrix $\mathbf{M}(\mathbf{q}_m^k, \omega)$. The discretisation interval is given by $\Delta t = \frac{t_f}{N}$. The vectors $\mathbf{q}_d \in \mathbb{R}^f$, $\mathbf{u}_d \in \mathbb{R}^f$ denote the desired end states. The vectors $\mathbf{w}(\mathbf{q}_m^k, \omega)$ and $\mathbf{b}(\mathbf{q}_m^k, \omega)$ represent the generalised force directions of respective blocking forces and the respective ordinary control forces, respectively. The variable vector $\mathbf{y}$ is composed of:

$$\begin{aligned} \mathbf{y} = & \{ \mathbf{u}_1^{k+1}, \dots, \mathbf{u}_f^{k+1}, \tau_1^k, \dots, \tau_s^k, \Gamma_{b1}^k, \dots, \Gamma_{bp}^k, \gamma_{br,1}^{k+1}, \dots, \gamma_{br,p}^{k+1}, \mathbf{N}_1^k, \dots, \mathbf{N}_p^k, t_f \}, \\ & k = 1, \dots, N. \end{aligned} \quad (3.176)$$

Given this discretisation scheme, the augmented Lagrangian function is constructed as follows:

$$\begin{aligned}
\mathcal{L}_A^{k+1} = & t_f + (\max \mathbf{L}_T^{k+1})^2 - (\max \mathbf{L}_T^k)^2 + (\min \mathbf{L}_T^{k+1})^2 - (\min \mathbf{L}_T^k)^2 \\
& + \sum_{i=1}^N \sum_{j=1}^f (\mathbf{p}_B \mathbf{L}_{ij}^{k+1})^2 + (\mathbf{m}_B \mathbf{L}_{ij}^{k+1})^2 - (\mathbf{p}_B \mathbf{L}_{ij}^k)^2 - (\mathbf{m}_B \mathbf{L}_{ij}^k)^2 \\
& + \sum_{i=1}^N \sum_{j=1}^m (\mathbf{p}_R \mathbf{L}_{ij}^{k+1})^2 + (\mathbf{m}_R \mathbf{L}_{ij}^{k+1})^2 - (\mathbf{p}_R \mathbf{L}_{ij}^k)^2 - (\mathbf{m}_R \mathbf{L}_{ij}^k)^2 \\
& + \sum_{i=1}^N \sum_{j=1}^m (\mathbf{p}_L \mathbf{L}_{ij}^{k+1})^2 + (\mathbf{m}_L \mathbf{L}_{ij}^{k+1})^2 - (\mathbf{p}_L \mathbf{L}_{ij}^k)^2 - (\mathbf{m}_L \mathbf{L}_{ij}^k)^2 \\
& + \sum_{i=1}^N \sum_{j=1}^m (\mathbf{p}_{NR} \mathbf{L}_{ij}^{k+1})^2 + (\mathbf{m}_{NR} \mathbf{L}_{ij}^{k+1})^2 - (\mathbf{p}_{NR} \mathbf{L}_{ij}^k)^2 - (\mathbf{m}_{NR} \mathbf{L}_{ij}^k)^2 \\
& + \sum_{i=1}^N \sum_{j=1}^m (\mathbf{p}_{NL} \mathbf{L}_{ij}^{k+1})^2 + (\mathbf{m}_{NL} \mathbf{L}_{ij}^{k+1})^2 - (\mathbf{p}_{NL} \mathbf{L}_{ij}^k)^2 - (\mathbf{m}_{NL} \mathbf{L}_{ij}^k)^2 \\
& + \sum_{j=1}^s (\tau_{\max} \mathbf{L}_{ij}^{k+1})^2 - (\tau_{\max} \mathbf{L}_{ij}^k)^2 + (\tau_{\min} \mathbf{L}_{ij}^{k+1})^2 - (\tau_{\min} \mathbf{L}_{ij}^k)^2 \\
& + \sum_{j=1}^{2f} (\mathbf{p}_f \mathbf{L}_j^{k+1})^2 + (\mathbf{m}_f \mathbf{L}_j^{k+1})^2 - (\mathbf{p}_f \mathbf{L}_j^k)^2 - (\mathbf{m}_f \mathbf{L}_j^k)^2.
\end{aligned} \tag{3.177}$$

The dual multipliers that occur in (3.177) are updated by the rules as given below. The updating rules clearly indicate the pairing among various constraints and dual multiplier vectors. The Lagrange multipliers that are related to the discretised form of the measure-differential equations of motion are updated by rules stated in rules (3.178) and (3.179):

$$\mathbf{p}_B \mathbf{L}_{ij}^{k+1} = \text{proj}_{\mathbb{R}_0^+}(\mathbf{p}_B \mathbf{L}_{ij}^k - c^k \mathbf{B}_{i,j}), \text{ for } i = 1 \dots N, j = 1 \dots f, \tag{3.178}$$

$$\mathbf{m}_B \mathbf{L}_{ij}^{k+1} = \text{proj}_{\mathbb{R}_0^+}(\mathbf{m}_B \mathbf{L}_{ij}^k + c^k \mathbf{B}_{i,j}), \text{ for } i = 1 \dots N, j = 1 \dots f. \tag{3.179}$$

Since every equality constraint is decomposed in two inequality constraints, Lagrange multipliers with an superscript $\mathbf{p}$ belongs to the greater equal case and $\mathbf{m}$ belongs to the less equal case. In a similar way the update rules for box-constraints on the discretised Lebesgue-measurable controls is given in equations (3.180) and (3.181):

$$\tau_{\max} \mathbf{L}_{i,j}^{k+1} = \text{proj}_{\mathbb{R}_0^+}(\tau_{\max} \mathbf{L}_{i,j}^k + c^k (\tau_{i,j} - \tau_j^{\max})), \text{ for } i = 1 \dots N, j = 1 \dots s, \tag{3.180}$$

$$\tau_{\min} \mathbf{L}_{i,j}^{k+1} = \text{proj}_{\mathbb{R}_0^+}(\tau_{\min} \mathbf{L}_{i,j}^k + c^k (\tau_j^{\min} - \tau_{i,j})), \text{ for } i = 1 \dots N, j = 1 \dots s. \tag{3.181}$$

The Lagrange multipliers that belong to the boundary constraints and the bounds on final time are updated by the rules as stated in equations (3.182) to (3.185):

$${}^p_f \mathbf{L}_j^{k+1} = \text{proj}_{\mathbb{R}_0^+}({}^p_f \mathbf{L}_j^k - c^k V_j), \text{ for } j = 1 \dots 2f, \quad (3.182)$$

$${}^m_f \mathbf{L}_j^{k+1} = \text{proj}_{\mathbb{R}_0^+}({}^m_f \mathbf{L}_j^k + c^k V_j), \text{ for } j = 1 \dots 2f, \quad (3.183)$$

$$\max \mathbf{L}_T^{k+1} = \text{proj}_{\mathbb{R}_0^+}(\max \mathbf{L}_T^k + c^k (t_{\min} - t_f)), \quad (3.184)$$

$$\min \mathbf{L}_T^{k+1} = \text{proj}_{\mathbb{R}_0^+}(\min \mathbf{L}_T^k + c^k (t_f - t_{\max})). \quad (3.185)$$

The Lagrange multipliers that belong to the Fischer-Burmeister functions are updated by the rules stated in equations (3.186) and (3.187):

$${}^p_\chi \mathbf{L}_{i,j}^{k+1} = \text{proj}_{\mathbb{R}_0^+}({}^p_\chi \mathbf{L}_{i,j}^k - c^k {}_\chi \Phi_{i,j}), \text{ for } i = 1 \dots N, j = 1 \dots p, \quad (3.186)$$

$${}^m_\chi \mathbf{L}_{i,j}^{k+1} = \text{proj}_{\mathbb{R}_0^+}({}^m_\chi \mathbf{L}_{i,j}^k + c^k {}_\chi \Phi_{i,j}), \text{ for } i = 1 \dots N, j = 1 \dots p, \quad (3.187)$$

for $\chi \in \mathcal{I} = \{L, R, NL, NR\}$.

3.3.2 Formulas for the Determination of the Numerical Gradient of the Augmented Lagrangian

In order to implement the gradient based minimisation of the augmented Lagrangian as given in equation (3.177) the gradient or set of gradients are analytically determined for the variables given in (3.176).

Gradient of $\mathcal{L}_A^k$ with respect to $\mathbf{u}_{l,n}$

The gradient of the augmented Lagrangian with respect to all discretised generalised velocities at every discretisation point and for every degrees of freedom is given by the expression (3.188):

$$\begin{aligned} \partial_{\mathbf{u}_{l,n}} \mathcal{L}_j^k &= \sum_{j=1}^f \sum_{i=1}^N (\partial_{\mathbf{u}_{l,n}} \mathbf{B}_{i,j} + \partial_{\mathbf{q}_{l,i}} \mathbf{B}_{i,j} \nabla_{\mathbf{u}_{l,n}} \mathbf{q}_{l,i}) ({}^p_B \mathbf{L}_{i,j}^{k+1} - {}^m_B \mathbf{L}_{i,j}^{k+1}) \\ &+ \sum_{j=1}^p \sum_{i=1}^N (\partial_{\mathbf{u}_{l,n}} ({}_L \Phi_{i,j}) + \partial_{\mathbf{q}_{l,i}} ({}_L \Phi_{i,j}) \nabla_{\mathbf{u}_{l,n}} \mathbf{q}_{l,i}) {}^p_L \mathbf{L}_{i,j}^{k+1} \\ &- \sum_{j=1}^p \sum_{i=1}^N (\partial_{\mathbf{u}_{l,n}} (-{}_L \Phi_{i,j}) + \partial_{\mathbf{q}_{l,i}} (-{}_L \Phi_{i,j}) \nabla_{\mathbf{u}_{l,n}} \mathbf{q}_{l,i}) {}^m_L \mathbf{L}_{i,j}^{k+1} \\ &+ \sum_{j=1}^p \sum_{i=1}^N (\partial_{\mathbf{u}_{l,n}} ({}_{NL} \Phi_{i,j}) + \partial_{\mathbf{q}_{l,i}} ({}_{NL} \Phi_{i,j}) \nabla_{\mathbf{u}_{l,n}} \mathbf{q}_{l,i}) {}^p_{NL} \mathbf{L}_{i,j}^{k+1} \\ &- \sum_{j=1}^p \sum_{i=1}^N (\partial_{\mathbf{u}_{l,n}} (-{}_{NL} \Phi_{i,j}) + \partial_{\mathbf{q}_{l,i}} (-{}_{NL} \Phi_{i,j}) \nabla_{\mathbf{u}_{l,n}} \mathbf{q}_{l,i}) {}^m_{NL} \mathbf{L}_{i,j}^{k+1} \\ &+ \partial_{\mathbf{q}_{N+1}^l} \mathbf{V}_l \nabla_{\mathbf{u}_{l,n}} \mathbf{q}_{N+1}^l ({}^m_f \mathbf{L}_l^{k+1} - {}^p_f \mathbf{L}_l^{k+1}), \quad n = 2 \dots N+1, \quad l = 1 \dots f. \end{aligned} \quad (3.188)$$

Gradient of $\mathcal{L}_A^k$ with respect to final time t_f

The gradient of the augmented Lagrangian with respect to the final time is given by the expression (3.189):

$$\begin{aligned}
\partial_{t_f} \mathcal{L}^k &= \sum_{j=1}^f \sum_{i=1}^N \left(\partial_{t_f} \mathbf{B}_{i,j} + \sum_{l=1}^f \partial_{\mathbf{q}_{l,i}} \mathbf{B}_{i,j} \nabla_{t_f} \mathbf{q}_{l,i} \right) (\mathbf{L}_{i,j}^{k+1} - \mathbf{L}_{i,j}^{k+1}) \\
&+ \sum_{j=1}^p \sum_{i=1}^N \sum_{l=1}^f \partial_{\mathbf{q}_{l,i}} (\mathbf{L} \Phi_{i,j}) \nabla_{t_f} \mathbf{q}_{l,i} \mathbf{L}_{i,j}^{k+1} - \partial_{\mathbf{q}_{l,i}} (-\mathbf{L} \Phi_{i,j}) \nabla_{t_f} \mathbf{q}_{l,i} \mathbf{L}_{i,j}^{k+1} \\
&+ \sum_{j=1}^p \sum_{i=1}^N \sum_{l=1}^f \partial_{\mathbf{q}_{l,i}} (\mathbf{NL} \Phi_{i,j}) \nabla_{t_f} \mathbf{q}_{l,i} \mathbf{L}_{i,j}^{k+1} - \partial_{\mathbf{q}_{l,i}} (-\mathbf{NL} \Phi_{i,j}) \nabla_{t_f} \mathbf{q}_{l,i} \mathbf{L}_{i,j}^{k+1} \\
&+ \sum_{j=1}^f \partial_{\mathbf{q}_{N+1}^j} (\mathbf{V}_j) \nabla_{t_f} \mathbf{q}_{N+1}^j (\mathbf{L}_j^{k+1} - \mathbf{L}_j^{k+1}).
\end{aligned} \tag{3.189}$$

Gradient of $\mathcal{L}_A^k$ with respect to discretised Lebesgue-measurable controls $\tau_{i,l}$

The gradient of the augmented Lagrangian with respect to the Lebesgue measurable controls $\tau_{i,l}$ at every discretisation point and for every controls is given by the expression (3.190):

$$\begin{aligned}
\partial_{\tau_{i,l}} \mathcal{L}^k &= \sum_{j=1}^f \partial_{\tau_{i,l}} \mathbf{B}_{i,j} (\mathbf{L}_{i,j}^{k+1} - \mathbf{L}_{i,j}^{k+1}) + (\tau_{min} \mathbf{L}_{i,l}^k - \tau_{max} \mathbf{L}_{i,l}^k), \\
l &= 1 \dots s, \quad i = 1 \dots N.
\end{aligned} \tag{3.190}$$

Gradient of $\mathcal{L}_A^k$ with respect to the discretised differential measure of blocking controls $\Gamma_{i,l}$

The gradient of the augmented Lagrangian with respect to the discretised differential-measure of the blocking controls $\Gamma_{i,l}$ at every discretisation interval and for every blocking controls is given by the expression (3.191):

$$\begin{aligned}
\partial_{\Gamma_{i,l}} \mathcal{L}^k &= \sum_{j=1}^f \partial_{\Gamma_{i,l}} \mathbf{B}_{i,j} (\mathbf{L}_{i,j}^{k+1} - \mathbf{L}_{i,j}^{k+1}) \\
&+ \sum_{j=1}^p \partial_{\Gamma_{i,l}} (\mathbf{L} \Phi_{i,j}) \mathbf{L}_{i,j}^{k+1} - \partial_{\Gamma_{i,l}} (-\mathbf{L} \Phi_{i,j}) \mathbf{L}_{i,j}^{k+1} + \partial_{\Gamma_{i,l}} (\mathbf{R} \Phi_{i,j}) \mathbf{L}_{i,j}^{k+1} - \partial_{\Gamma_{i,l}} (-\mathbf{R} \Phi_{i,j}) \mathbf{L}_{i,j}^{k+1} \\
l &= 1 \dots p, \quad i = 1 \dots N.
\end{aligned} \tag{3.191}$$

Gradient of $\mathcal{L}_A^k$ with respect to the discretised differential measure of controls $\mathbf{N}_{i,l}$

The gradient of the augmented Lagrangian with respect to the discretised differential-measure of the blocking controls $\mathbf{N}_{i,l}$ at every discretisation interval and for every blocking controls is given by the expression (3.192):

$$\partial_{N_{i,l}} \mathcal{L}^k = \sum_{\forall \chi \in \mathcal{I}} \sum_{j=1}^p \partial_{N_{i,l}} (\chi \Phi_{i,j})_{\chi}^p \mathbf{L}_{i,j}^{k+1} - \partial_{N_{i,l}} (-\chi \Phi_{i,j})_{\chi}^m \mathbf{L}_{i,j}^{k+1}, \quad l = 1 \dots p \quad i = 1 \dots N \quad (3.192)$$

for $\chi \in \mathcal{I} = \{L, R, NL, NR\}$.

Gradient of $\mathcal{L}_A^k$ with respect to the $\gamma_{br,l}^{i+1}$

The gradient of the augmented Lagrangian with respect to the discretised slack velocity $\gamma_{br,l}^{i+1}$ at every discretisation point and for every blockable direction is given by the expression (3.193):

$$\partial_{\gamma_{br,l}^{i+1}} \mathcal{L}^k = \sum_{\forall \chi \in \mathcal{I}} \sum_{j=1}^p \partial_{\gamma_{br,l}^{i+1}} (\chi \Phi_{i,j})_{\chi}^p \mathbf{L}_{i,j}^{k+1} - \partial_{\gamma_{br,l}^{i+1}} (-\chi \Phi_{i,j})_{\chi}^m \mathbf{L}_{i,j}^{k+1}, \quad l = 1 \dots p, \quad i = 1 \dots N \quad (3.193)$$

for $\chi \in \mathcal{I} = \{L, R, NL, NR\}$.

3.3.3 An Augmented Lagrangian Based Nonlinear Programming Algorithm

The algorithm consists of three loops which are embedded in each other as shown in figure 3.6. The most outer iteration which is the main iteration is controlling the level of nonlinearity of the system. The continuation parameter ω is screwed slowly upwards so that when ω becomes one the system of equalities and inequalities represents the discretised mechanical system, and if ω is zero the system resembles a linear mechanical system. The intermediate iteration performs for each given ω a number of successive minimisations of the resulting augmented Lagrangian functional $\mathcal{L}_a^k$ and updates after every minimisation the Lagrange multiplier vectors L according to relations from (3.178) to (3.187). The inner iteration performs the minimisation of the augmented Lagrangian $\mathcal{L}_a^k$ at stage k for a given ω and Lagrange multiplier vector $\mathbf{L}^k$ by a modified conjugate gradients method as shown in flowchart 3.7. The formulas for the augmented Lagrangian functional, update of Lagrange multipliers and the subgradient of the augmented Lagrangian are given in the preceding section.

The conjugate gradient algorithm does not use prespecified conjugate directions, but instead computes the directions as the algorithm progresses. At each stage of the algorithm, the direction is calculated as a linear combination of the previous direction and the current gradient, in a such a way that all the directions are mutually $\mathbf{Q}$ -conjugate. This method makes use of the fact that for a quadratic function of n variables, one can locate the function minimizer

by performing at most n steps along mutually conjugate directions. Shortly the algorithm is described for the quadratic case in the sequel. Given a QP of the form:

$$f(\mathbf{y}) = \frac{1}{2} \langle \mathbf{y}, \mathbf{Q} \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{b} \rangle, \quad (3.194)$$

for $\mathbf{y} \in \mathbb{R}^n$ and a positive definitive and symmetric matrix $\mathbf{Q}$, the first direction of search at point $\mathbf{y}^1$ is in the direction of steepest descent, which is given by:

$$\mathbf{d}^1 = -\nabla f(\mathbf{y}^1). \quad (3.195)$$

The next point is obtained as:

$$\mathbf{y}^1 = \mathbf{y}^1 + \kappa_0 \mathbf{d}^1 \quad (3.196)$$

where κ_0 is obtained as a result of an optimisation

$$\kappa_0 = \arg \min_{\kappa_0 \geq 0} f(\mathbf{y}^1 + \kappa \mathbf{d}^1) = -\frac{\langle \mathbf{g}^1, \mathbf{d}^1 \rangle}{\langle \mathbf{d}^1, \mathbf{Q} \mathbf{d}^1 \rangle}. \quad (3.197)$$

In the next stage, a direction $\mathbf{d}^1$ is searched that is $\mathbf{Q}$ -conjugate to $\mathbf{d}^1$. In general, at step $k+1$ the search direction is chosen as:

$$\mathbf{d}^{k+1} = -\mathbf{g}^{k+1} + \eta_k \mathbf{d}^k, \quad k = 0, 1, 2, \dots \quad (3.198)$$

In order to obtain $\mathbf{Q}$ -conjugate search directions, η_k is chosen as:

$$\eta_k = \frac{\langle \mathbf{g}^{k+1}, \mathbf{Q} \mathbf{d}^k \rangle}{\langle \mathbf{d}^k, \mathbf{Q} \mathbf{d}^k \rangle}. \quad (3.199)$$

In the case of nonquadratic minimisation a conjugate gradient algorithm that do not require explicit knowledge of the Hessian matrix $\mathbf{Q}$ is desirable. The modifications of the conjugate gradient algorithm for a function for which the Hessian is unknown but in which objective function values and gradients are available is given in the following. The modifications are all based on algebraically manipulating the formula for η_k in such a way that $\mathbf{Q}$ is eliminated. There are three well-known methods as enlisted below for the update of η_k :

- Hestenes-Stiefel formula

$$\eta_k = \frac{\langle \mathbf{g}^{k+1}, \mathbf{g}^{k+1} - \mathbf{g}^k \rangle}{\langle \mathbf{d}^k, \mathbf{g}^{k+1} - \mathbf{g}^k \rangle}, \quad (3.200)$$

- Polak-Ribière formula

$$\eta_k = \frac{\langle \mathbf{g}^{k+1}, \mathbf{g}^{k+1} - \mathbf{g}^k \rangle}{\langle \mathbf{g}^k, \mathbf{g}^k \rangle}, \quad (3.201)$$

- Fletcher-Reeves formula

$$\eta_k = \frac{\langle \mathbf{g}^{k+1}, \mathbf{g}^{k+1} \rangle}{\langle \mathbf{g}^k, \mathbf{g}^k \rangle}. \quad (3.202)$$

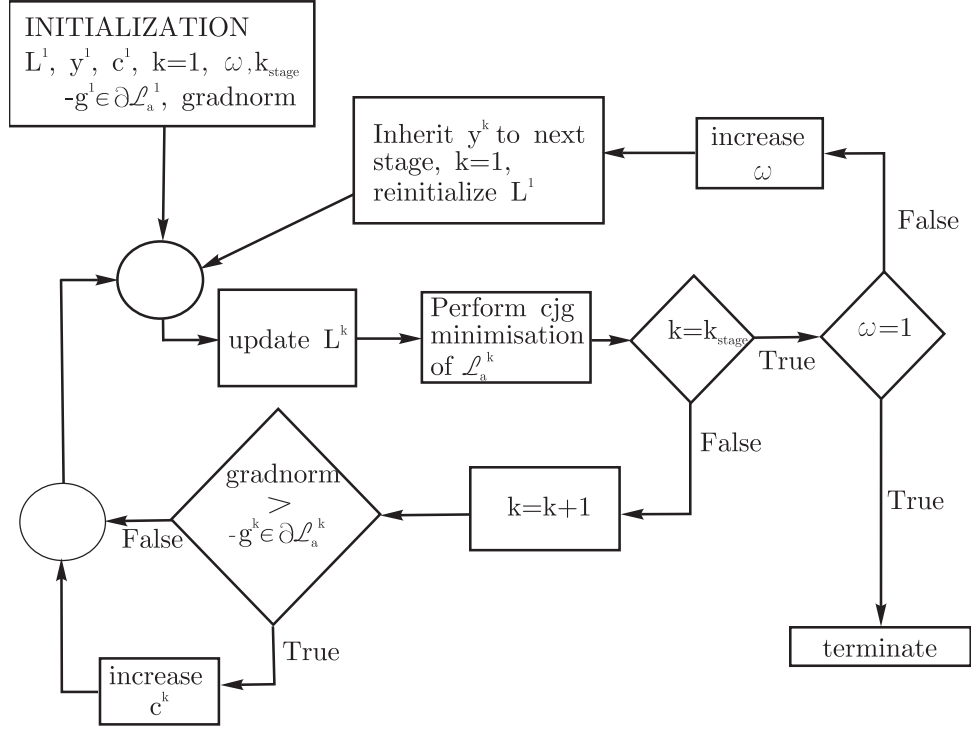


Figure 3.6: The simplified flowchart of the algorithm.

These modified conjugate gradient strategies rely on estimating the Hessian of a nonquadratic goal function and requires updating of the search direction to the exact gradient every certain number of steps. It is not clear which of these methods is superior since their performance is quite problem specific. In the case where the Hessian is not determined explicitly, the optimal value of κ can not be given in a closed form as in equation (3.197). The step given in equation (3.196) is replaced by a line search algorithm. The line search method in determining the minimum in the direction of search has been an interpolation based on bisection line search.

The termination criteria for the intermediate iterations take advantage of a property of the augmented Lagrangian method which is called partial minimisation. The minimisations between successive Lagrange multiplier updates is not performed exactly but instead partially so that the slow convergence in the vicinity of the intermediate minima does not pose a severe problem to the run times. The number of minimisations performed for each value of ω are increased linearly proportional as it approaches one. The number of steps for the minimisation of each augmented Lagrangian is also increased to the square of the corresponding ω -stage.

The primal variables of the optimisation have been inherited to the next ω -stage. The dual values are also inherited to the next stage except at initial stages of the optimisation.

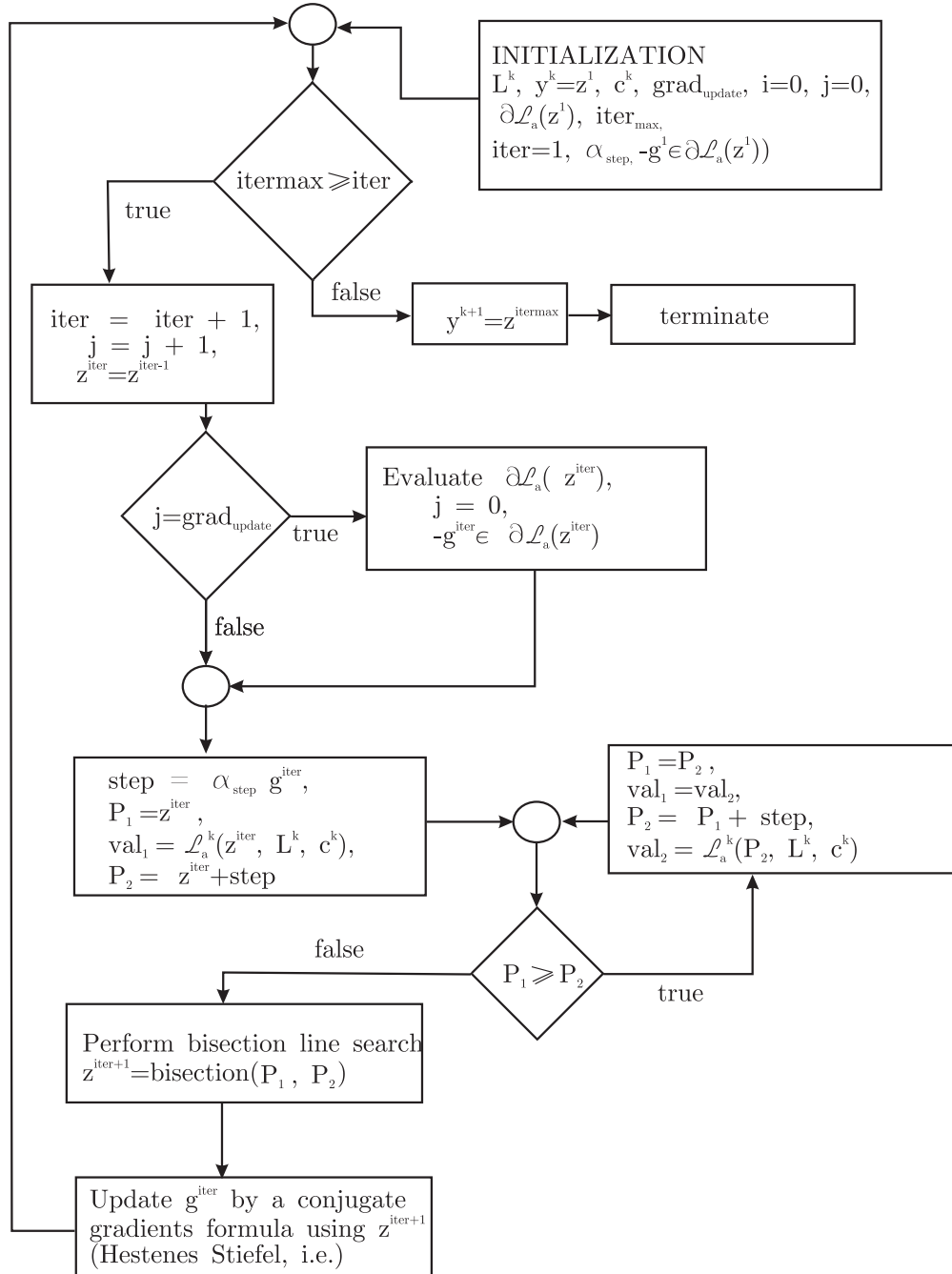


Figure 3.7: The flowchart of the conjugate gradients minimisation.

3.3.4 Implementation of the Algorithm

The algorithm has been implemented with MATLAB. The numerically intensive and time consuming parts has been programmed in C and by the generation of mex-files, these C codes are appended to the MATLAB main routine. The ANSI C codes have reduced the run times extensively. Following tasks in the optimisation have been performed with mex-files:

- The function evaluations of the augmented Lagrangian $\mathcal{L}_a$,
- The interpolations base on bisection line search,
- The update of Lagrange multipliers,
- The evaluation of the exact gradient of the augmented Lagrangian function,
- The conjugate gradients minimisation of the augmented Lagrangian function.

The programming approach based on combining MATLAB and ANSI C codes provides flexibility to perform modifications easily whereas the runtimes are drastically reduced by codes.

In the design of the structure of the code modularity has been a prime issue in order to enable easy adaptation of the code to other optimisation problems. This has been achieved by characterizing each optimisation by its unique gradient and augmented Lagrangian function. Easy interchangeability of gradient and augmented Lagrangian modules are achieved.

3.4 Numerical Example

In this section a numerical example for the algorithm that is explained in section 3.3 is presented. The investigated mechanical system is shown in figure 3.8. It has two rotational degrees of freedom denoted by α and β . The DOF α is measured in the positive sense from the ${}_1\mathbf{e}_x$ axis of the inertial frame I , which is located at the point O . The DOF β is measured in the positive sense from the ${}_1\mathbf{e}^x$ axis of the inertial frame I as shown in figure 3.8. The link 1 rotates around point O and link 2 rotates around point A which is attached to link 1. The motor at link O drives the system by the Lebesgue measurable control force τ , which has upper and lower limits denoted by $\tau_{\min}$ and $\tau_{\max}$, respectively. The brake at A connected to the links one and two, has two operating modes, either it allows totally free motion or it blocks such that the relative angular velocity between links one and two given by $\dot{\alpha} - \dot{\beta}$ reduces immediately to zero. This requires depending on the pre-blocking state of the system a jump in the relative angular velocity, which only can be induced by impulsive forces. The center of masses of the links 1 and 2 are denoted by S_1 and S_2 , respectively. The moments of inertia of links 1 and 2 with respect to the their center of masses are given by Θ_1 and Θ_2 , respectively. The masses of the links are denoted by m_1 and m_2 , respectively and the lengths of each link is given by l_1

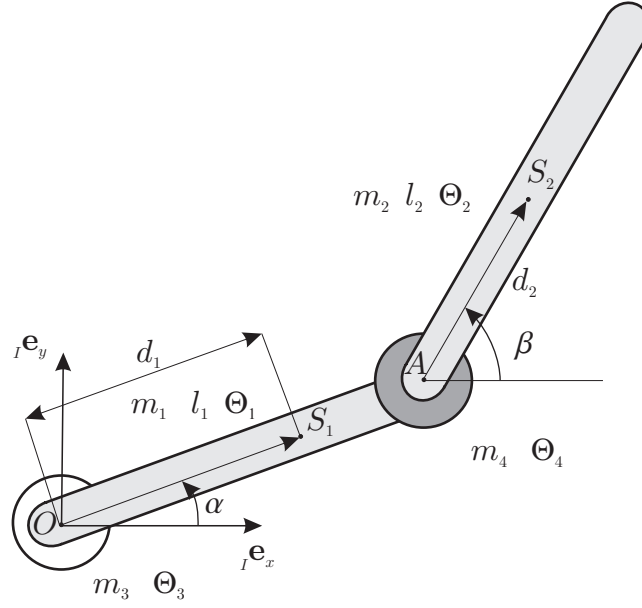


Figure 3.8: The parameters of the planar double pendulum.

and l_2 . The center of mass of the brake is located at A and the mass and moment of inertia of the brake with respect to A are given by m_4 and Θ_4 , respectively. Similarly, the motor, that is located at O has mass m_3 and moment of inertia Θ_3 with respect to O . The numerical values d_1 and d_2 denote the distances $\|OS_1\|$ and $\|AS_2\|$, respectively. The system consists of two rigid bodies. The first rigid body consists of link one, the motor, and the part of the brake that moves with link one. The second rigid body consists of link two and the part of the brake that moves together with link two. Accordingly, the physical properties of mass and inertia of the rigid bodies one and two, as well as positions of the respective center of masses r_{cm1} and r_{cm2} depend on the physical properties of the constituent components. With respect to the number of degrees of freedom, the system has two operating modes. If the brake does not block, the system is a double pendulum in the plane, and if the brake hinders relative motion of the second link with respect to the first link then the system has one rotational degree of freedom. Indeed, the system has infinitely many modes in the one degree of freedom case depending at which relative degree $\alpha - \beta$ the underactuated link is kept blocked. In the sequel, two time-optimal trajectories are investigated and presented. Since blocking is an highly dissipative process as presented in subsection 2.6.3, it is suitable in the context of time-optimal control then any other goal function such as control-effort optimal control. The numerical values of various parameters in numerical case study is given in table 3.1. The equality and inequality constraints of the

m_1 [kg]	m_2 [kg]	m_3 [kg]	m_4 [kg]	Θ_1 [kg m ²]	Θ_2 [kg m ²]	Θ_3 [kg m ²]	Θ_4 [kg m ²]	l_1 [m]	l_2 [m]	$ \tau $ [nm]
1	1	1	1	0.05	0.05	0.05	0.05	1	1	$< 2 $
m_{41} [kg]	m_{42} [kg]	d_1 [m]	d_2 [m]	m_1 [kg m ²]	m_2 [kg m ²]	m_3 [kg m ²]	V_1 [—]	V_2 [—]	r_{cm1} [m]	r_{cm2} [m]
0.5	0.5	$\frac{l_1}{2}$	$\frac{l_2}{2}$	1.16	$\frac{1}{9}$	$\frac{1}{3}$	0.2755	2.069	0.4	$\frac{1}{3}$

Table 3.1: System parameters of the manipulator for cases A, B, C.

Inequality	Dual Multiplier	Inequality	Dual Multiplier
$\Phi_R \preceq \mathbf{0}$	${}^p\mathbf{L}_R$	$-\Phi_R \preceq \mathbf{0}$	${}^m\mathbf{L}_R$
$\Phi_L \preceq \mathbf{0}$	${}^p\mathbf{L}_L$	$-\Phi_L \preceq \mathbf{0}$	${}^m\mathbf{L}_L$
$\Phi_{NL} \preceq \mathbf{0}$	${}^p\mathbf{L}_{NL}$	$-\Phi_{NL} \preceq \mathbf{0}$	${}^m\mathbf{L}_{NL}$
$\Phi_{NR} \preceq \mathbf{0}$	${}^p\mathbf{L}_{NR}$	$-\Phi_{NR} \preceq \mathbf{0}$	${}^m\mathbf{L}_{NR}$
$\mathbf{B}_\alpha \preceq \mathbf{0}$	${}^p_B\mathbf{L}_\alpha$	$-\mathbf{B}_\alpha \preceq \mathbf{0}$	${}^m_B\mathbf{L}_\alpha$
$\mathbf{B}_\beta \preceq \mathbf{0}$	${}^p_B\mathbf{L}_\beta$	$-\mathbf{B}_\beta \preceq \mathbf{0}$	${}^m_B\mathbf{L}_\beta$
$\boldsymbol{\tau} - \boldsymbol{\tau}_{\max} \preceq \mathbf{0}$	${}^{\max}\mathbf{L}_\tau$	$\boldsymbol{\tau}_{\min} - \boldsymbol{\tau} \preceq \mathbf{0}$	${}^{\min}\mathbf{L}_\tau$
$\beta^{N+1} - \beta_f \leq 0$	${}^pL_{\beta f}$	$\beta_f - \beta^{N+1} \leq 0$	${}^mL_{\beta f}$
$\alpha^{N+1} - \alpha_f \leq 0$	${}^pL_{\alpha f}$	$\alpha_f - \alpha^{N+1} \leq 0$	${}^mL_{\alpha f}$
$\dot{\beta}^{N+1} - \dot{\beta}_f \leq 0$	${}^pL_{\dot{\beta} f}$	$\dot{\beta}_f - \dot{\beta}^{N+1} \leq 0$	${}^mL_{\dot{\beta} f}$
$\dot{\alpha}^{N+1} - \dot{\alpha}_f \leq 0$	${}^pL_{\dot{\alpha} f}$	$\dot{\alpha}_f - \dot{\alpha}^{N+1} \leq 0$	${}^mL_{\dot{\alpha} f}$
$t_f - t_{\max} \leq 0$	${}^{\max}L_{tf}$	$t_{\min} - t_f \leq 0$	${}^{\min}L_{tf}$

Table 3.2: Correspondence between inequalities and dual multipliers in the NLP of the planar double pendulum.

NLP are given below for $k = 1 \dots N$, in relations (3.203) to (3.216):

$$(\dot{\alpha}[k+1] - \dot{\alpha}[k]) + \omega V_1 \cos(\alpha_m[k] - \beta_m[k])(\dot{\beta}[k+1] - \dot{\beta}[k]) \quad (3.203)$$

$$+ \omega V_1 \Delta t \sin(\alpha_m[k] - \beta_m[k]) \dot{\beta}^2[k] - \tau[k] \Delta t + \Gamma[k] = 0, \quad (3.204)$$

$$(\dot{\beta}[k+1] - \dot{\beta}[k]) + \omega V_2 \cos(\alpha_m[k] - \beta_m[k])(\dot{\alpha}[k+1] - \dot{\alpha}[k]) \quad (3.204)$$

$$- \omega V_2 \Delta t \sin(\alpha_m[k] - \beta_m[k]) \dot{\alpha}^2[k] - \Gamma[k] = 0, \quad (3.205)$$

$$\Phi_{FB} ((\Gamma[k] + \mathbf{N}[k]), \gamma_{br}[k+1]) = 0, \quad (3.205)$$

$$\Phi_{FB} ((-\Gamma[k] + \mathbf{N}[k]), \gamma_{br}[k+1] + \dot{\alpha}[k+1] - \dot{\beta}[k+1]) = 0, \quad (3.206)$$

$$\Phi_{FB} (\gamma_{br}[k+1], \mathbf{N}[k]) = 0, \quad (3.207)$$

$$\Phi_{FB} (\gamma_{br}[k+1] + \dot{\alpha}[k+1] - \dot{\beta}[k+1], \mathbf{N}[k]) = 0, \quad (3.208)$$

$$\tau[k] - \tau_{\max} \leq 0, \quad (3.209)$$

$$\tau_{\max} - \tau[k] \leq 0, \quad (3.210)$$

$$t_f - t_{\max} \leq 0, \quad (3.211)$$

$$t_{\min} - t_f \leq 0, \quad (3.212)$$

$$\dot{\alpha}[N+1] - \dot{\alpha}_f = 0, \quad (3.213)$$

$$\dot{\beta}[N+1] - \dot{\beta}_f = 0, \quad (3.214)$$

$$\alpha[N+1] - \alpha_f = 0, \quad (3.215)$$

$$\beta[N+1] - \beta_f = 0. \quad (3.216)$$

In the sequel equations (3.203) and (3.204) are denoted by $\mathbf{B}_\alpha[k]$ and $\mathbf{B}_\beta[k]$, respectively. The corresponding dual multipliers are shown in table 3.2. The center of masses, equivalent inertias and some other parameters are formulated in dependence on the physical properties of the constituent components as below:

$$\begin{aligned} r_{cm1} &= l_1 \frac{(\frac{m_1}{2} + m_{41})}{m_{41} + m_1 + m_3}, \\ r_{cm2} &= m_2 \frac{l_2}{2(m_{42} + m_2)}, \\ \Theta_{cm1} &= \Theta_1 + \Theta_3 + \Theta_{41} + m_1 \left(\frac{l_1}{2} - r_{cm1}\right)^2 + m_3 r_{cm1}^2 + m_{41} (l_1 - r_{cm1})^2, \\ \Theta_{cm2} &= \Theta_{42} + \Theta_2 + m_2 \left(\frac{l_2}{2} - r_{cm2}\right)^2 + m_{42} r_{cm2}^2, \\ m_1 &= m_1 r_{cm1}^2 + m_2 l_1^2, \\ m_2 &= m_2 r_{cm2}^2, \\ m_3 &= m_2 l_1 r_{cm2}, \\ V_1 &= \frac{m_3}{m_1 + \Theta_1}, \\ V_2 &= \frac{m_3}{m_2 + \Theta_2}. \end{aligned}$$

3.4.1 Case A

In this maneuver the aim is to reach the final position given by $\alpha_f = \frac{3\pi}{4}$, $\beta_f = \pi$, $\dot{\alpha}_f = 0$, $\dot{\beta}_f = 0$ time-optimally.

The numerical information on the maneuver A is given in table 3.4. The problem is discretized over a number of 201 points. In this time-optimal maneuver the system starts in the blocked state. The whole maneuver takes 3.395 seconds. There are eight transitions at times $\mathbf{t}[27] = 0.46$ s, $\mathbf{t}[34] = 0.58$ s, $\mathbf{t}[89] = 1.51$ s, $\mathbf{t}[158] = 2.68$ s, $\mathbf{t}[159] = 2.7$ s, $\mathbf{t}[199] = 3.38$ s. At $\mathbf{t}[27]$ the underactuated link is released. At transition interval $\mathbf{P}[34]$, the underactuated link is blocked impactively. The relative velocity $\gamma_{br}[34]$ is decreased by $1.60 \frac{\text{rad}}{\text{sec}}$ to $-2.3927 \cdot 10^{-4} \frac{\text{rad}}{\text{sec}}$. The discretised differential measure of the normal control force $\mathbf{N}[34]$ that is applied in $\mathbf{P}[34]$ is 18.98 N m sec and impulsive control force that is applied to brake down the passive link amounts to $\mathbf{\Gamma}[34] = 10.88$ N m sec. The passive link is again released at $\mathbf{t}[89] = 1.51$ s, and is impactively blocked during interval $\mathbf{P}[158]$ by $\mathbf{N}[158] = 34.50$ N m sec and impulsive control force that amounts to $\mathbf{\Gamma}[158] = -34.50$ N m sec. The relative velocity $\gamma_{br}[158]$ is increased by $0.34 \frac{\text{rad}}{\text{sec}}$ to $\gamma_{br}[159] = -1.5094 \cdot 10^{-3} \frac{\text{rad}}{\text{sec}}$. At transition time $\mathbf{t}[159] = 2.7$ s the passive link is released and is impactively blocked during interval $\mathbf{P}[199]$ by $\mathbf{N}[199] = 37.95$ N m sec and impulsive control force that amounts to $\mathbf{\Gamma}[199] = 37.96$ N m sec. The relative velocity $\gamma_{br}[199]$ is decreased by $3.31 \frac{\text{rad}}{\text{sec}}$ to $\gamma_{br}[200] = 0.0035 \frac{\text{rad}}{\text{sec}}$. The mode sequence and the duration of each mode is given in table 3.3. The action of the double pendulum is seen in figure 3.10. Qualitatively, the motor accelerates the system in the blocked phase and throws the underactuated link at $\mathbf{t}[89] = 1.51$ s, by releasing the second link, and switching to maximal deceleration, in order to increase the relative angle as fast as possible. This serves two goals; namely; it reduces the equivalent moment of inertia that resists the motor in moving the system to the desired end state, and brings the second link closer to the desired relative angle of 135 degrees. The motor accelerates the system again. The final stage of the maneuver starts with the impulsive blocking and releasing action at $\mathbf{t}[158] = 2.68$ s during interval $\mathbf{P}[158]$. The relative angle overshoots 135 degrees, and the system this by making use of the dynamic coupling between the actuated and underactuated link. The impulsive blocking at $\mathbf{t}[158] = 2.68$ s serves to stop the overshooting motion by reducing the relative velocity and acceleration to zero as can be seen in figures 3.9 and 3.10. Figure 3.11 depicts the evolution of the optimal motor-torque history and the discretised measures of normal contact force and frictional braking moment. Figure 3.12 depicts the numerical residues of equations $\frac{\mathbf{B}_\alpha[k]}{\Delta t}$ and $\frac{\mathbf{B}_\beta[k]}{\Delta t}$, respectively. The residues indicate the level of accuracy which is attained at the solution. Figure 3.13 presents the residues of the four reformulation functions. The actual values of the complementarities are shown in figure 3.18. The dual multipliers for various inequalities as listed in table 3.3 are listed in figures 3.14, 3.16 and 3.17. The evolution of the costate dynamics has been obtained by a post optimisation calculation and are shown in figure 3.15.

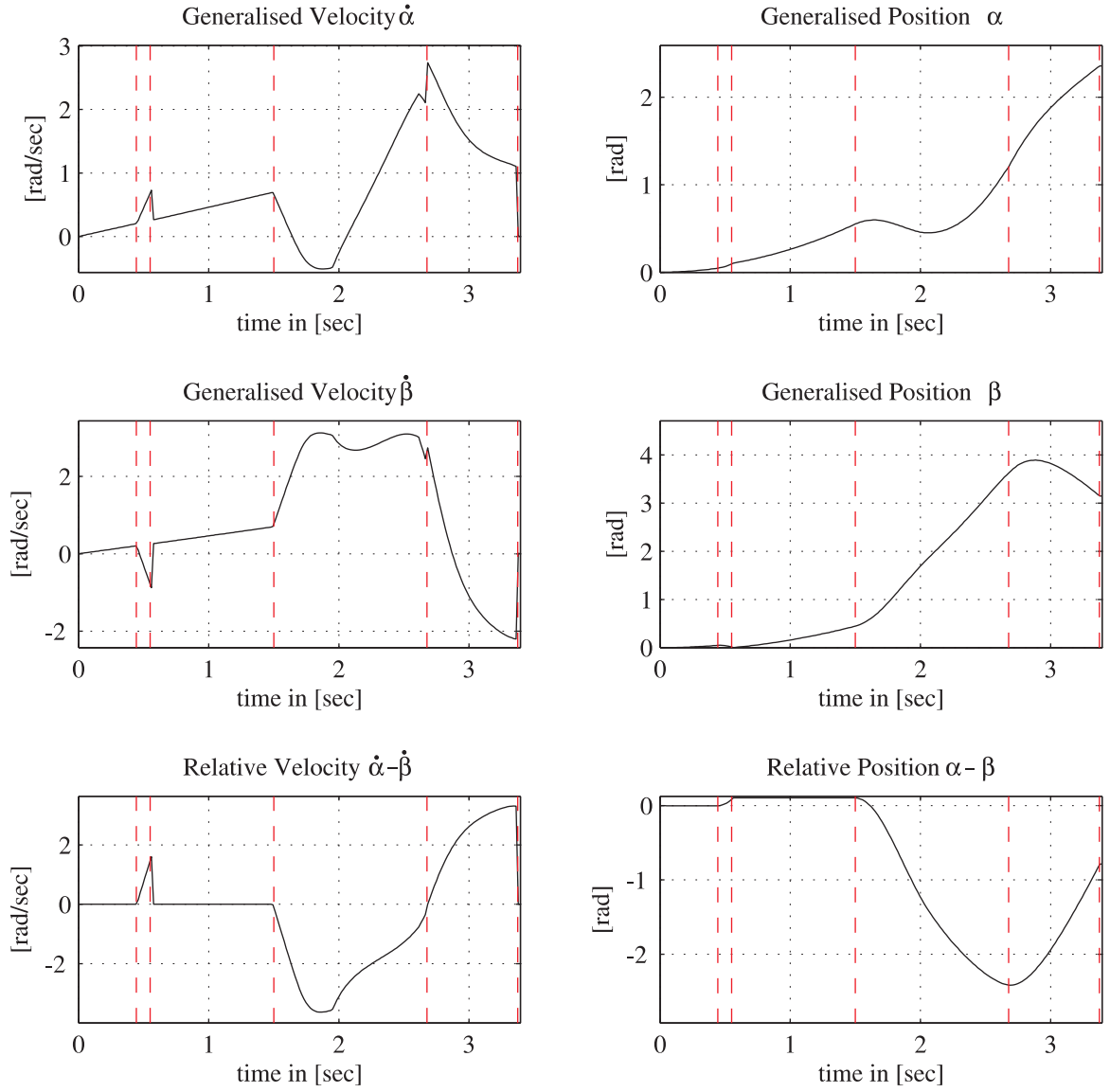


Figure 3.9: Case A: The optimal evolution of the generalised velocities $\dot{\alpha}$, $\dot{\beta}$, relative velocity $\dot{\alpha} - \dot{\beta}$, generalize positions α , β and relative position $\alpha - \beta$. (red lines mark the transition times)

Modes	1-DOF	2-DOF	1-DOF	2-DOF	1-DOF	2-DOF	1-DOF
Duration	0.458 s	0.12 s	0.93 s	1.17 s	0.017 s	0.68 s	0.017 s
Intervals	1-27	27-34	34-89	89-158	158-159	159-199	199-200

Table 3.3: Case A: The mode sequence, duration and intervals of modes.

# of disc. points	200	penalty c	$1e^5$	# of primal variables	1201
# of dual variables	2810	increment in ω	0.1	k_{stage}	40
# of complementarities	800	α_0 [rad]	0	$\dot{\alpha}_0$ [rad/sec]	0
α_f [rad]	$\frac{3\pi}{4}$	$\dot{\alpha}_f$ [rad/sec]	0	β_0 [rad]	0
$\dot{\beta}_0$ [rad/sec]	0	β_f [rad]	π	$\dot{\beta}_f$ [rad/sec]	0

Table 3.4: Case A: Optimisation parameters, initial and final states of the optimisation.

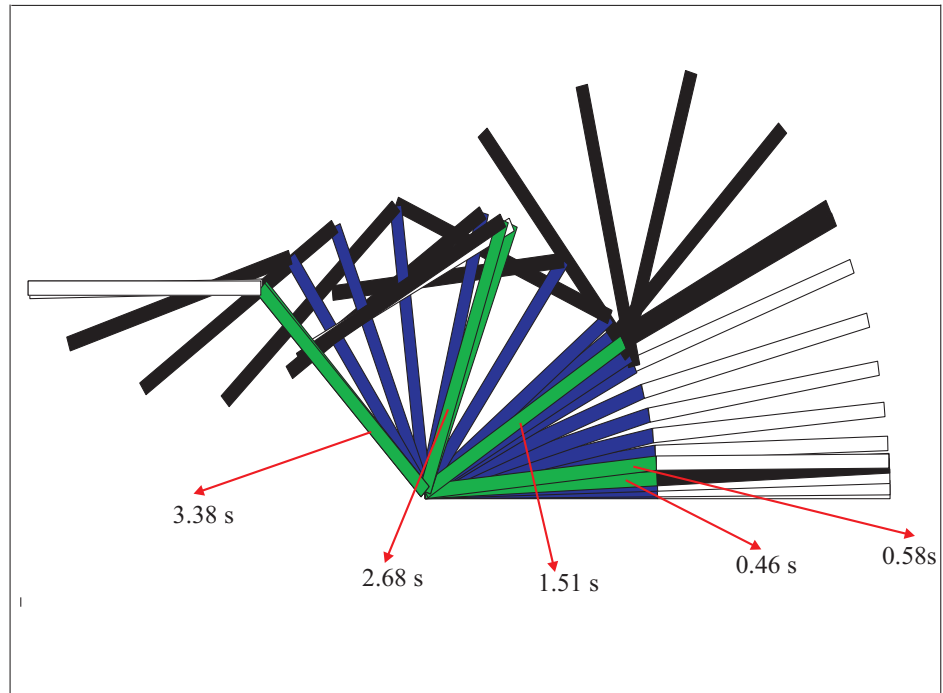


Figure 3.10: Case A: Maneuver of the double pendulum (Color code: For link 2 white is blocked, black is unblocked, for link 1 green position at transition interval).

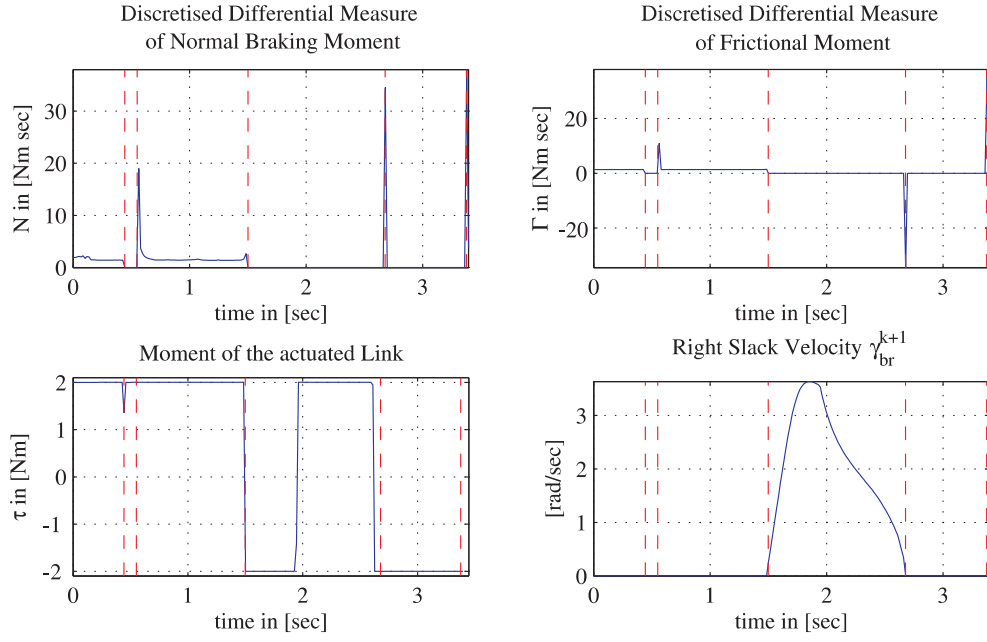


Figure 3.11: Case A: The optimal evolution of the discretised differential measures N , Γ , the torque τ and the slack velocity γ_{br} (red lines mark the transition times).

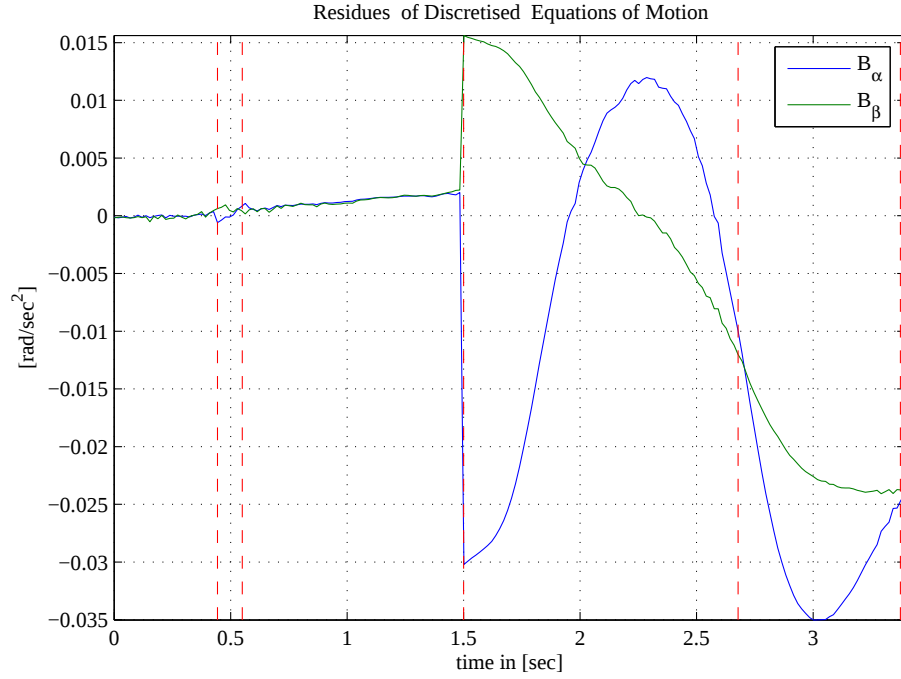


Figure 3.12: Case A: The numerical residue of the discretised equations of motion approx. by $\frac{\mathbf{B}[k]}{\Delta t}$ (red lines mark the transition times).

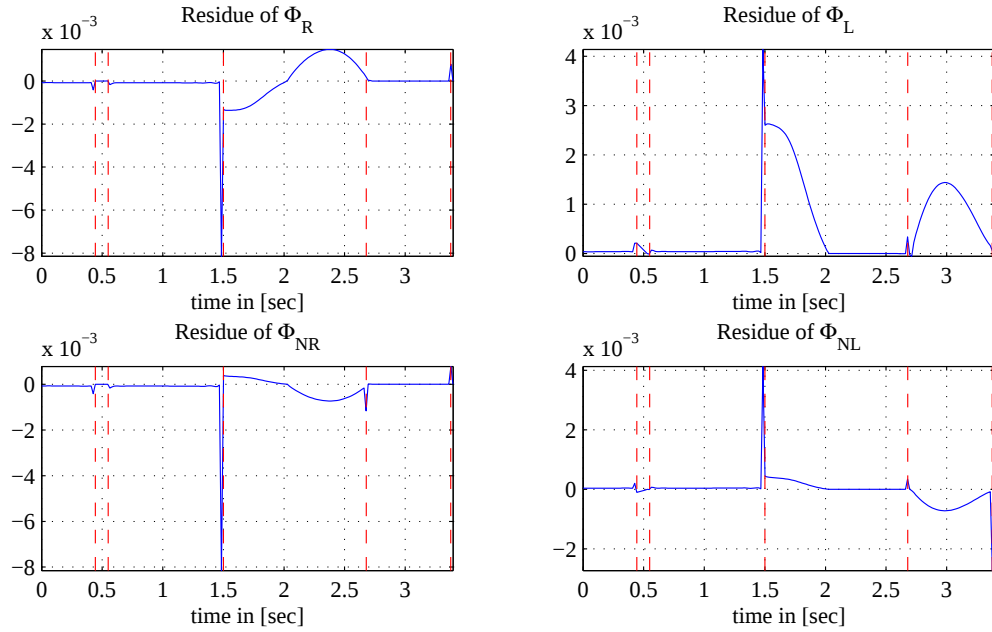


Figure 3.13: Case A: Residues of the reformulation functions Φ_R , Φ_L , Φ_{NR} and Φ_{NL} (red lines mark the transition times).

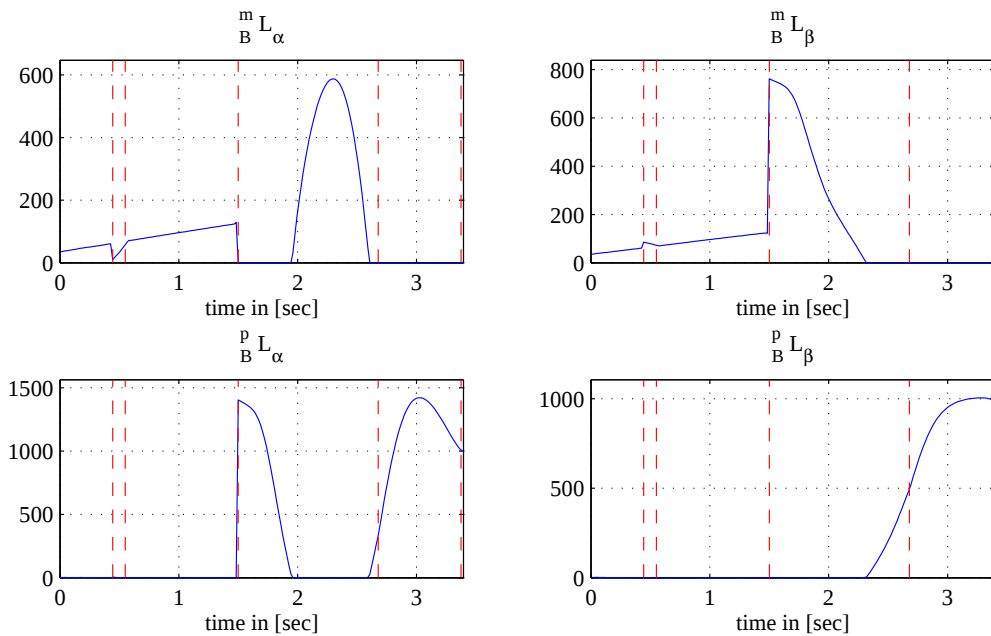


Figure 3.14: Case A: The optimal evolution of the dual multipliers ${}^m_B L_\alpha$, ${}^m_B L_\beta$, ${}^p_B L_\alpha$ and ${}^p_B L_\beta$ (red lines mark the transition times).

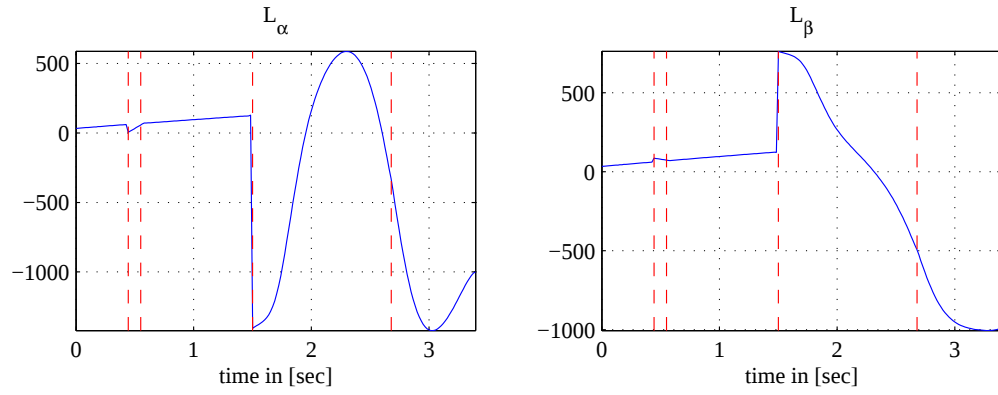


Figure 3.15: Case A: The time evolution of the costate dynamics in the second-order form (red lines mark the transition times).

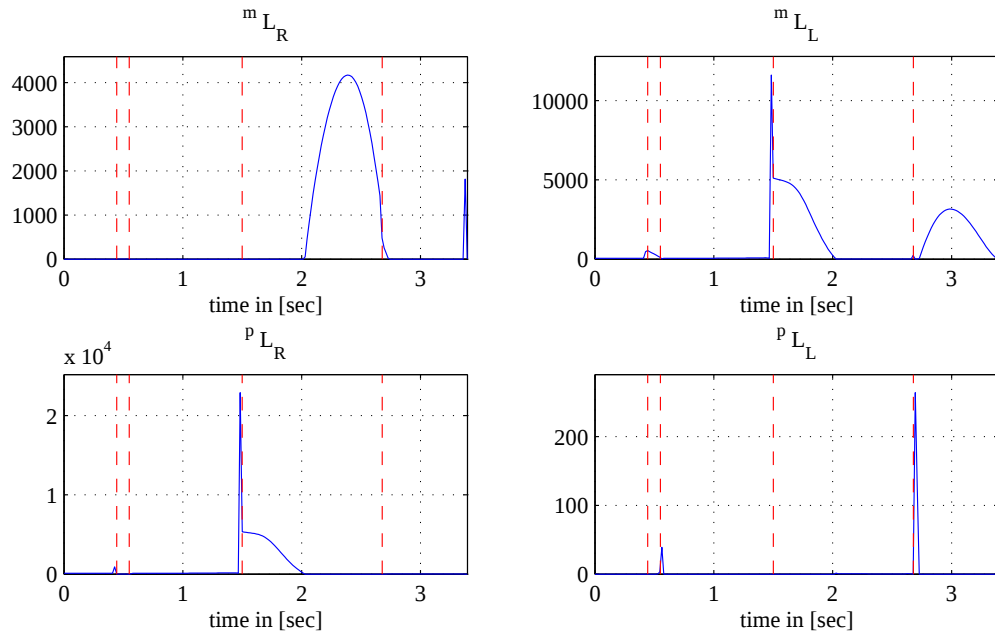


Figure 3.16: Case A: The optimal evolution of the dual multipliers mL_R , mL_L , pL_R and pL_L (red lines mark the transition times).

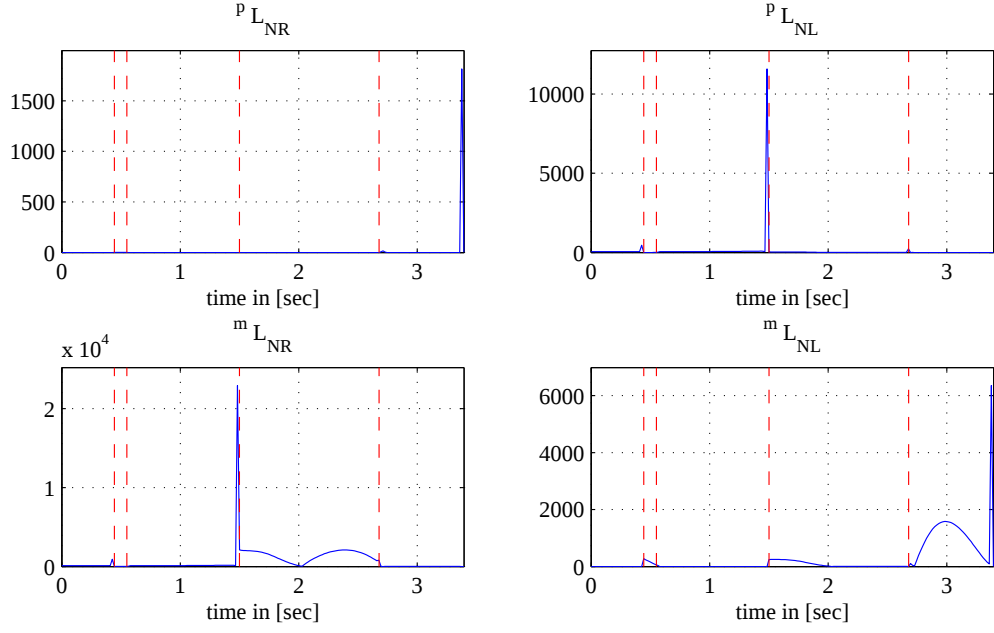


Figure 3.17: Case A: The optimal evolution of the dual multipliers ${}^mL_{NR}$, ${}^mL_{NL}$, ${}^pL_{NR}$ and ${}^pL_{NL}$ (red lines mark the transition times).

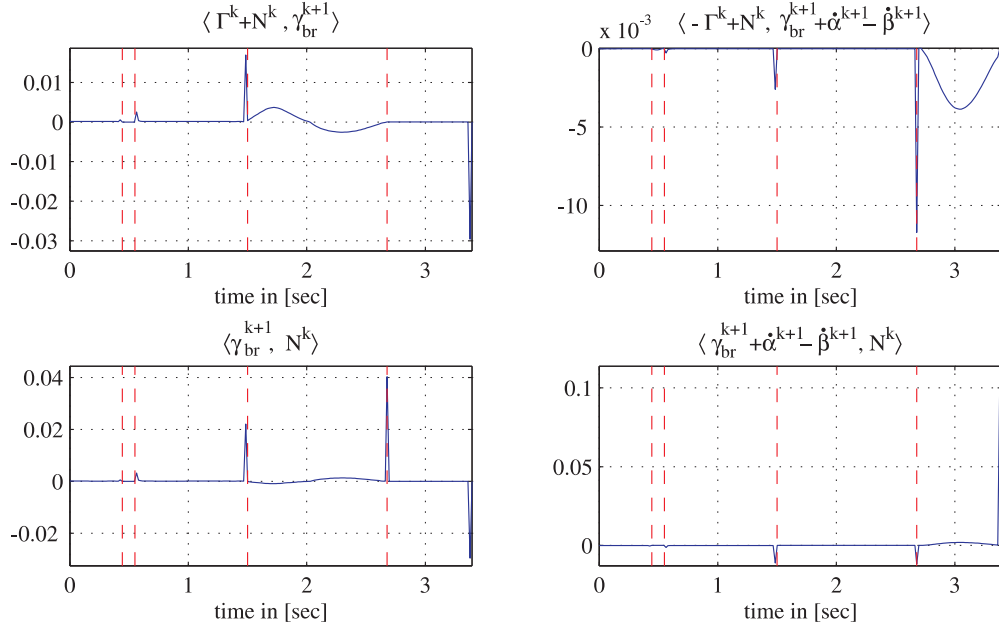


Figure 3.18: Case A: The residues of the complementarities (red lines mark the transition times).

Modes	1-DOF	2-DOF	1-DOF	2-DOF	1-DOF
Duration	0.41 s	0.68 s	0.75 s	0.57 s	2.11 s
Intervals	1-19	19-49	49-82	82-107	107-200

Table 3.5: Case B: The mode sequence, duration and intervals of modes.

3.4.2 Case B

In this maneuver the aim is to reach the final position given by $\alpha_f = \pi$, $\beta_f = \pi$, $\dot{\alpha}_f = 0$, $\dot{\beta}_f = 0$ time-optimally. The numerical information on the maneuver B is given in table 3.6. The problem is discretized over a number of 201 points. In this time-optimal maneuver the system starts in the blocked state. The whole maneuver takes 4.53 seconds. There are four transitions at times $\mathbf{t}[19] = 0.43$ s, $\mathbf{t}[49] = 1.11$ s, $\mathbf{t}[82] = 1.86$ s, $\mathbf{t}[107] = 2.42$ s. At $t[19]$ the underactuated link is released. At transition interval $\mathbf{P}[49]$, the underactuated link is blocked impactively. The relative velocity $\gamma_{br}[49]$ is decreased by $2.41 \frac{\text{rad}}{\text{sec}}$ to $3.6776 \cdot 10^{-4} \frac{\text{rad}}{\text{sec}}$. The discretised differential measure of the normal control force $\mathbf{N}[49]$ that is applied in $\mathbf{P}[49]$ is 62.00 N m sec and impulsive control force that is applied to brake down the passive link amounts to $\mathbf{\Gamma}[49] = 61.97$ N m sec. The passive link is again released at $\mathbf{t}[82] = 1.86$ s, and is impactively blocked during interval $\mathbf{P}[107]$ by $\mathbf{N}[107] = -34.88$ N m sec and impulsive control force that amounts to $\mathbf{\Gamma}[107] = -34.65$ N m sec. The relative velocity $\gamma_{br}[107]$ is increased by $7.55 \frac{\text{rad}}{\text{sec}}$ to $\gamma_{br}[107] = -6.0490 \cdot 10^{-4} \frac{\text{rad}}{\text{sec}}$. The mode sequence and the duration of each mode is given in table 3.5. The maneuver of the underactuated manipulator is seen in figure 3.20. Qualitatively, the motor accelerates the system with unblocked underactuated link. The relative angle decreases such that the equivalent moment of inertia, that resists the motor is reduced. The link two is then kept blocked between 1.11 s and 1.86 s. The release of link two enables to throw it to the desired final position. As the desired relative angle of 0 degrees is reached, the arm is blocked impulsively, and the motor decelerates the system to the final position in a bang-bang manner.

Figure 3.21 depicts the evolution of the optimal motor-torque history and the discretised measures of normal contact force and frictional braking moment. Figure 3.22 depicts the numerical residues of equations $\frac{\mathbf{B}_\alpha[k]}{\Delta t}$ and $\frac{\mathbf{B}_\beta[k]}{\Delta t}$, respectively. The residues indicate the level of accuracy which is attained at the solution. Figure 3.23 presents the residues of the four reformulation functions. The actual values of the complementarities are shown in figure 3.28. The dual multipliers for various inequalities as listed in table 3.3 are listed in figures 3.24, 3.26 and 3.27. The evolution of the costate dynamics has been obtained by a post optimisation calculation and are shown in figure 3.25.

# of disc. points	200	penalty c	$1e^5$	# of primal variables	1201
# of dual variables	2810	increment in ω	0.1	k_{stage}	40
# of complementarities	800	α_0 [rad]	0	$\dot{\alpha}_0$ [rad/sec]	0
α_f [rad]	π	$\dot{\alpha}_f$ [rad/sec]	0	β_0 [rad]	0
$\dot{\beta}_0$ [rad/sec]	0	β_f [rad]	π	$\dot{\beta}_f$ [rad/sec]	0

Table 3.6: Case B: Optimisation parameters, initial and final states of the optimisation.

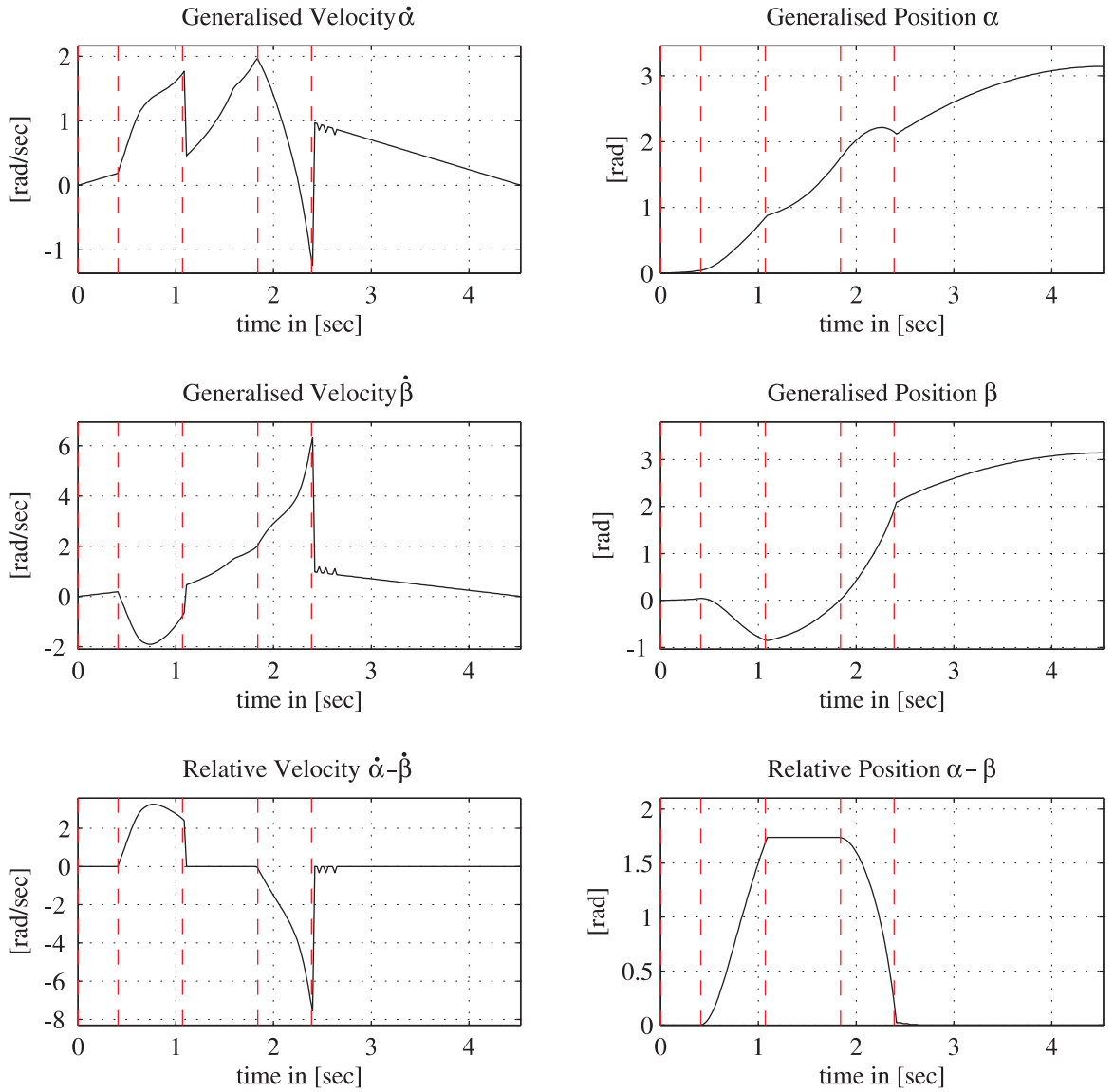


Figure 3.19: Case B: The optimal evolution of the generalised velocities $\dot{\alpha}$, $\dot{\beta}$, relative velocity $\dot{\alpha} - \dot{\beta}$, generalised positions α , β and relative position $\alpha - \beta$. red lines mark the transition times)

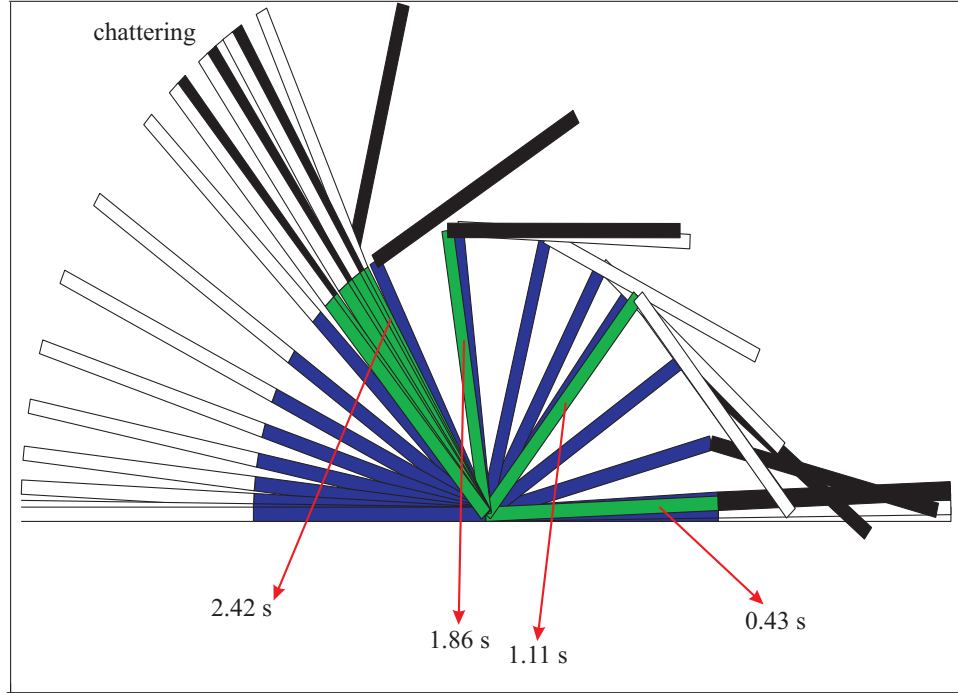


Figure 3.20: Case B: Maneuver of the double pendulum (Color code: For link 2 white is blocked, black is unblocked, for link 1 green position at transition interval).

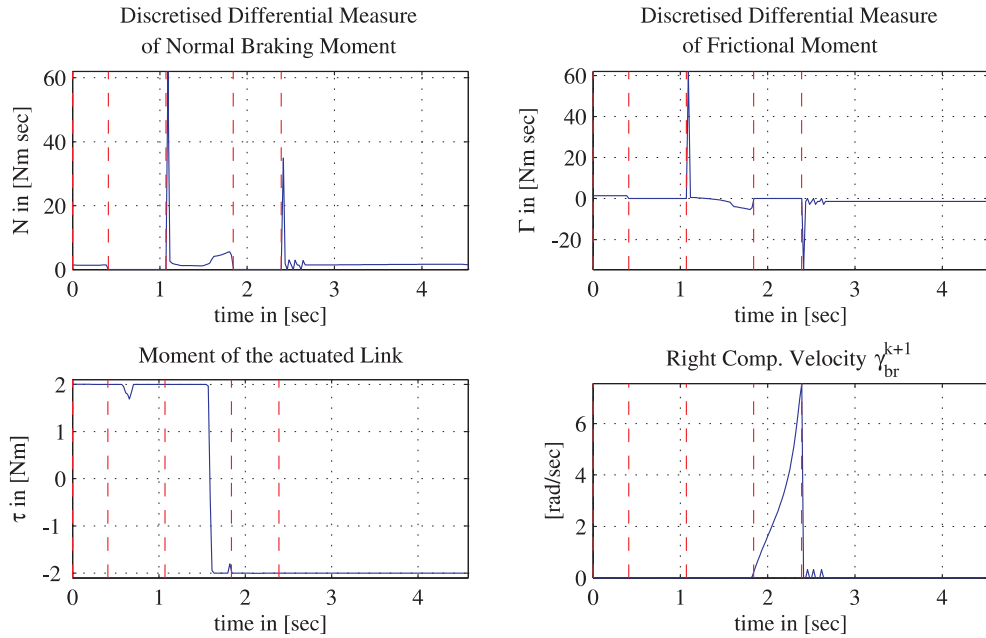


Figure 3.21: Case B: The optimal evolution of the discretised differential measures N , Γ , the torque τ and the slack velocity γ_{br} (red lines mark the transition times).

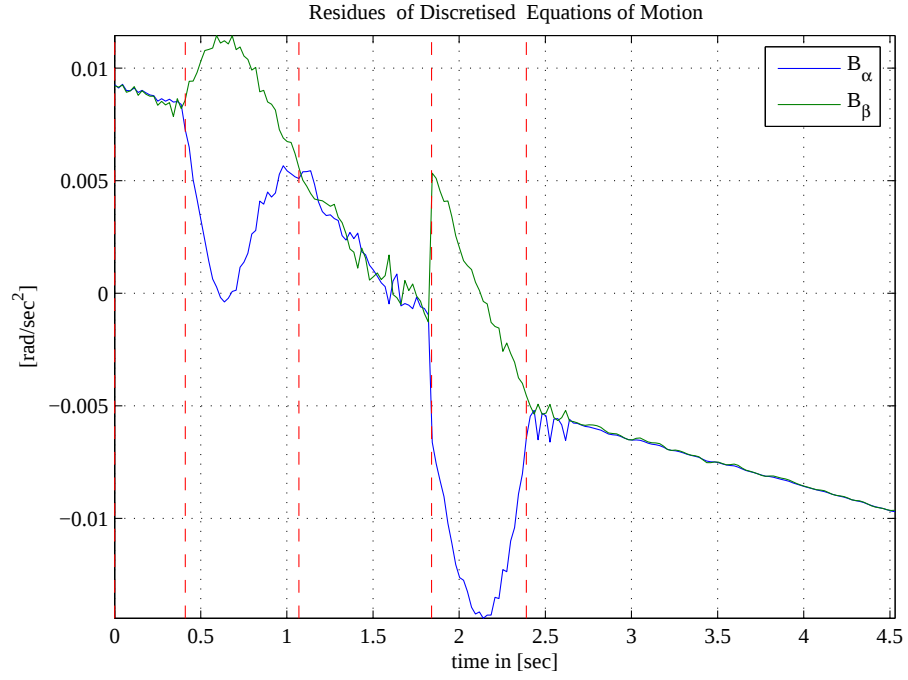


Figure 3.22: Case B: The numerical residue of the discretised equations of motion of motion approx. by $\frac{\mathbf{B}[k]}{\Delta t}$ (red lines mark the transition times).

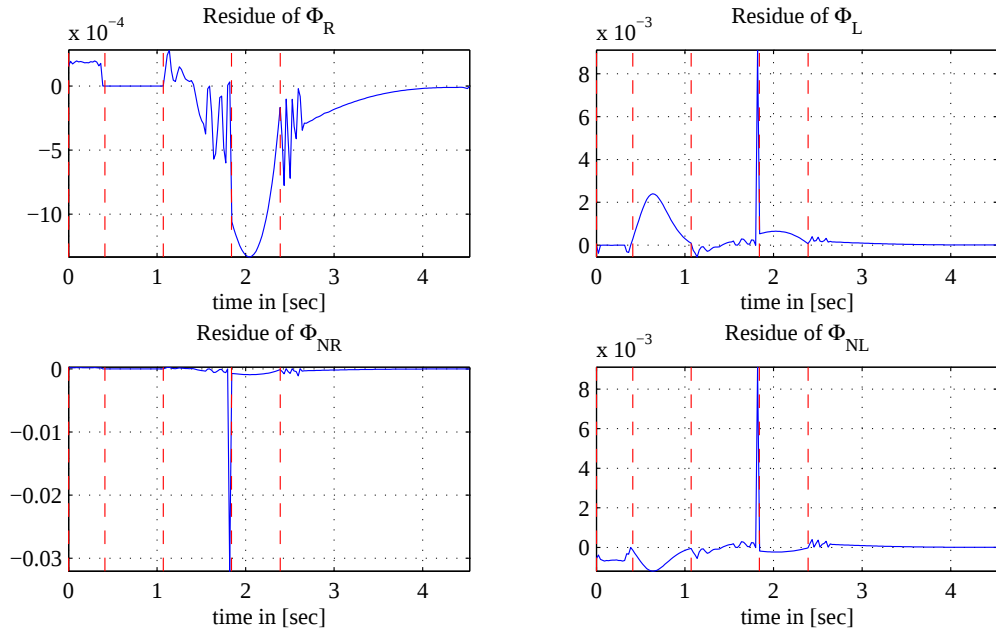


Figure 3.23: Case B: Residues of the reformulation functions Φ_R , Φ_L , Φ_{NR} and Φ_{NL} (red lines mark the transition times).

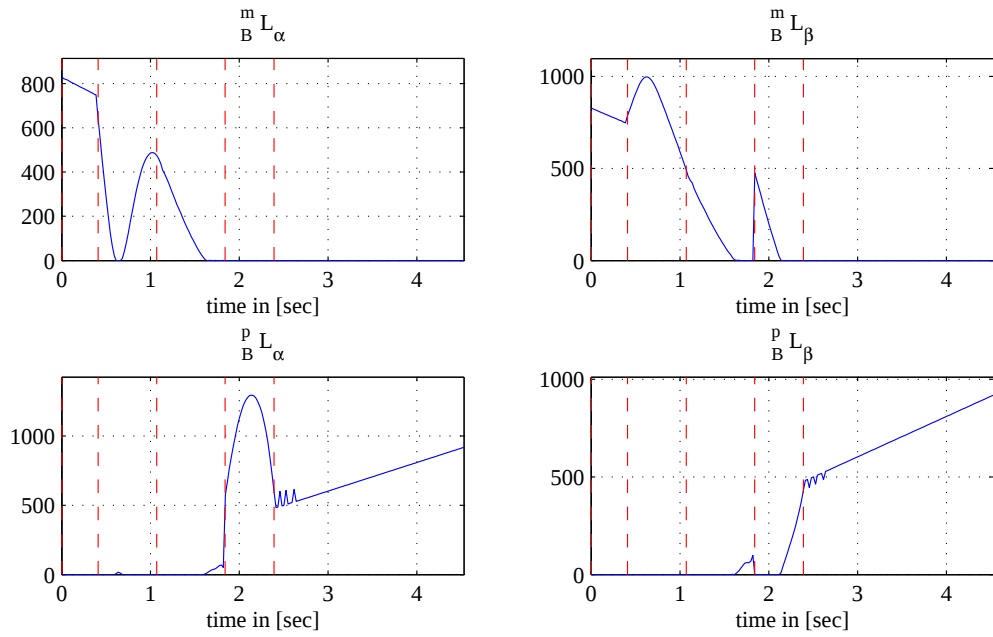


Figure 3.24: Case B: The optimal evolution of the dual multipliers ${}^m_B L_\alpha$, ${}^m_B L_\beta$, ${}^p_B L_\alpha$ and ${}^p_B L_\beta$ (red lines mark the transition times).

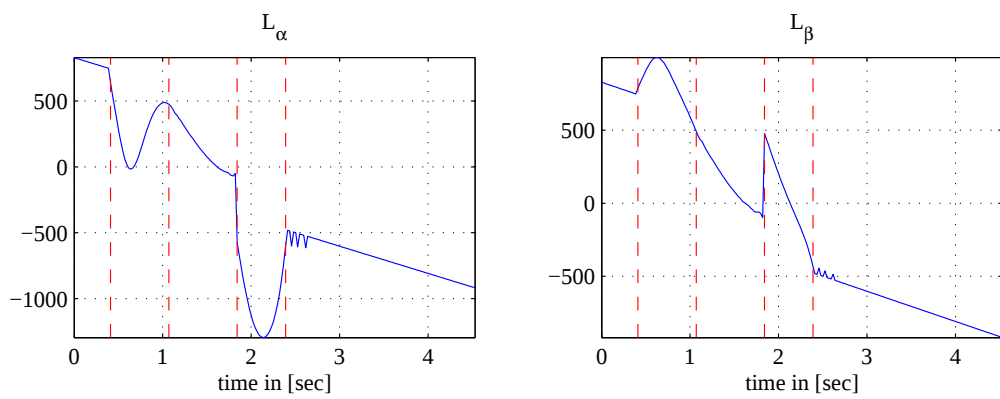


Figure 3.25: Case B: The time evolution of the costate dynamics in the second-order form (red lines mark the transition times).

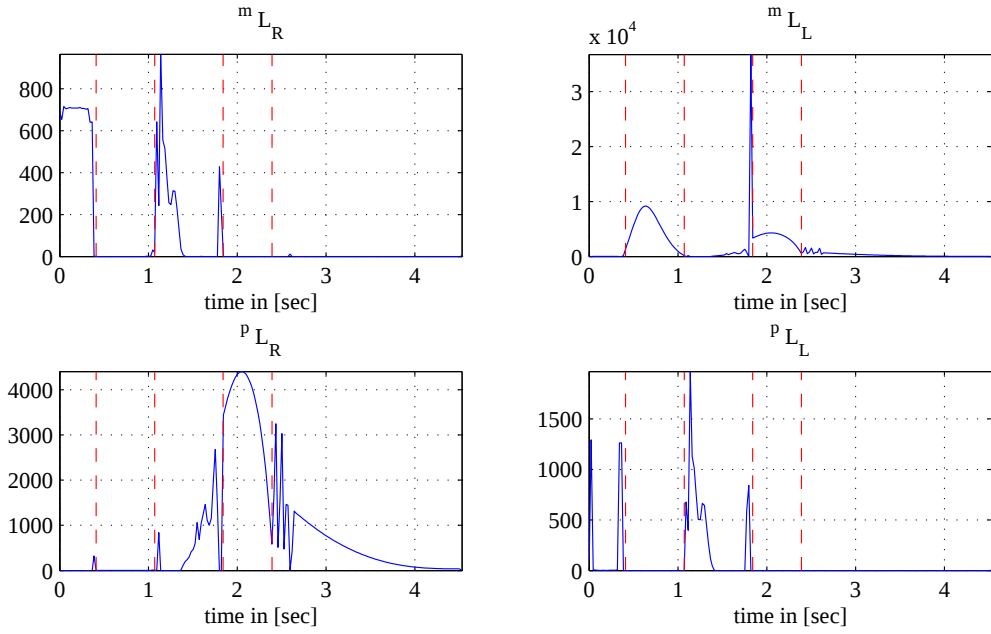


Figure 3.26: Case B: The optimal evolution of the dual multipliers ${}^m L_R$, ${}^m L_L$, ${}^p L_R$ and ${}^p L_L$ (red lines mark the transition times).

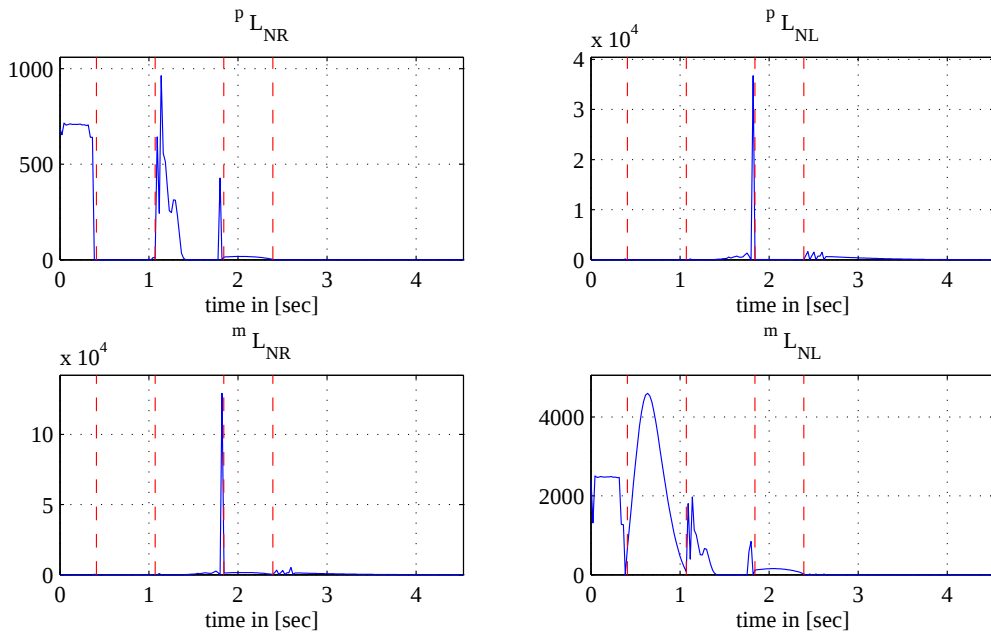


Figure 3.27: Case B: The optimal evolution of the dual multipliers ${}^m L_{NR}$, ${}^m L_{NL}$, ${}^p L_{NR}$ and ${}^p L_{NL}$ (red lines mark the transition times).

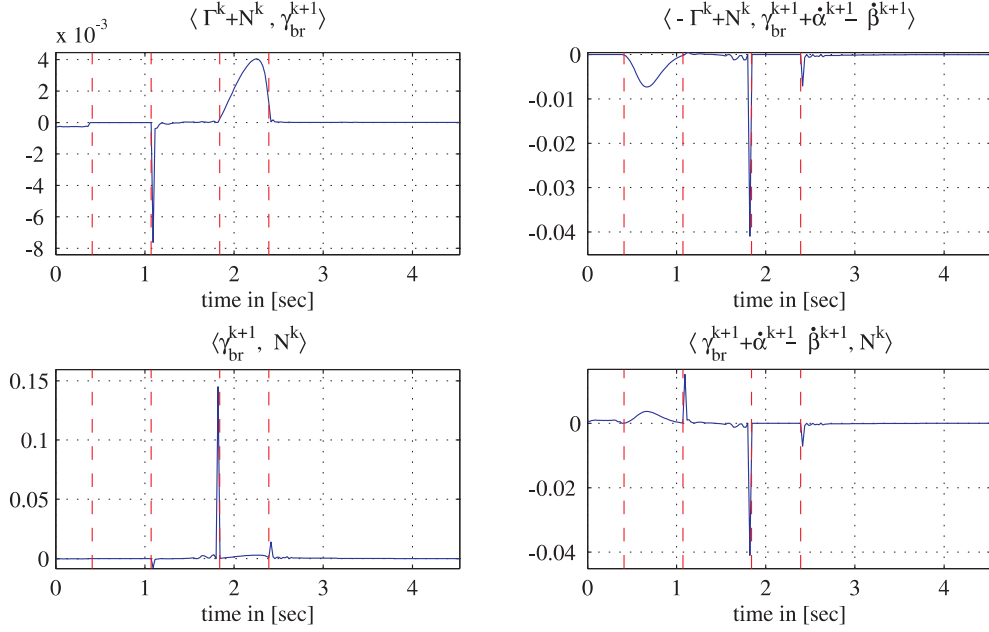


Figure 3.28: Case B: The residues of the complementarities (red lines mark the transition times).

3.4.3 Case C

In this maneuver the aim is to reach the final position given by $\alpha_f = \pi$, $\beta_f = \pi$, $\dot{\alpha}_f = 0$, $\dot{\beta}_f = 0$ time-optimally. The numerical parameters used in the optimisation are seen in table 3.8. In this time-optimal maneuver the system starts in the blocked state. The whole maneuver takes 2.77 seconds. There are four transitions at times $t[3] = 0.04155$ s, $t[96] = 1.3296$ s, $t[147] = 2.036$ s, $t[187] = 2.59$ s. At $t[3]$ the underactuated link is released. At transition interval $P[96]$, the underactuated link is blocked impactively. The relative velocity $\gamma_{br}[96]$ is reduced by $0.5 \frac{\text{rad}}{\text{sec}}$ to $-0.0225 \frac{\text{rad}}{\text{sec}}$. The discretized differential measure of the normal control force $N[96]$ that is applied in $P[96]$ is 38.64 N m sec and impulsive control force that is applied to brake down the passive link amounts to $\Gamma[96] = -38.64$ N m sec. The passive link is again released at $t[147] = 2.036$ s, and is impactively blocked during interval $P[187]$ by $N[187] = 73.71$ N m sec and impulsive control force that amounts to $\Gamma[187] = 73.71$ N m sec. The relative velocity $\gamma_{br}[187]$ is reduced by $2.0262 \frac{\text{rad}}{\text{sec}}$ to $\gamma_{br}[188] = 0.0022 \frac{\text{rad}}{\text{sec}}$. The mode sequence and the duration of each mode is given in table 3.7. Qualitatively, the maneuver can be divided into three parts. In the first part of the maneuver, the motor moves the first link in clockwise direction while the second link is unblocked. The relative angle is decreased as seen in figure 3.29, and the underactuated link is blocked at 1.32 seconds. The aim of the maneuver, is to block the second arm in a position so that the equivalent moment of inertia that resists the motor is minimised. In the second part of the maneuver, the motor rushes to the position $\alpha = \frac{\pi}{2}$ while the second

Modes	1-DOF	2-DOF	1-DOF	2-DOF	1-DOF
Duration	0.04 s	1.29 s	0.71 s	0.55 s	0.18 s
Intervals	1-3	3-96	96-147	147-187	187-200

Table 3.7: Case C: The mode sequence, duration and intervals of modes.

# of disc. points	200	penalty c	$1e^5$	# of primal variables	1201
# of dual variables	2810	increment in ω	0.1	k_{stage}	40
# of complementarities	800	α_0 [rad]	0	$\dot{\alpha}_0$ [rad/sec]	0
α_f [rad]	$\frac{\pi}{2}$	$\dot{\alpha}_f$ [rad/sec]	0	β_0 [rad]	0
$\dot{\beta}_0$ [rad/sec]	0	β_f [rad]	π	$\dot{\beta}_f$ [rad/sec]	

Table 3.8: Case C: Optimisation parameters, initial and final states of the optimisation.

link is blocked until 2.04 seconds. At this time instant, the underactuated link is released so that through the dynamic coupling. it moves to the end position $\beta = \pi$, where it is impactively blocked at the final time instant at 2.59 seconds.

Figure 3.31 depicts the evolution of the optimal motor-torque history and the discretised measures of normal contact force and frictional braking moment. Figure 3.32 depicts the numerical residues of equations $\frac{\mathbf{B}_\alpha[k]}{\Delta t}$ and $\frac{\mathbf{B}_\beta[k]}{\Delta t}$, respectively. The residues indicate the level of accuracy which is attained at the solution. Figure 3.33 presents the residues of the four reformulation functions. The actual values of the complementarities are shown in figure 3.38. The dual multipliers for various inequalities as listed in table 3.3 are listed in figures 3.34, 3.36 and 3.37. The evolution of the costate dynamics has been obtained by a post optimisation calculation and are shown in figure 3.35.

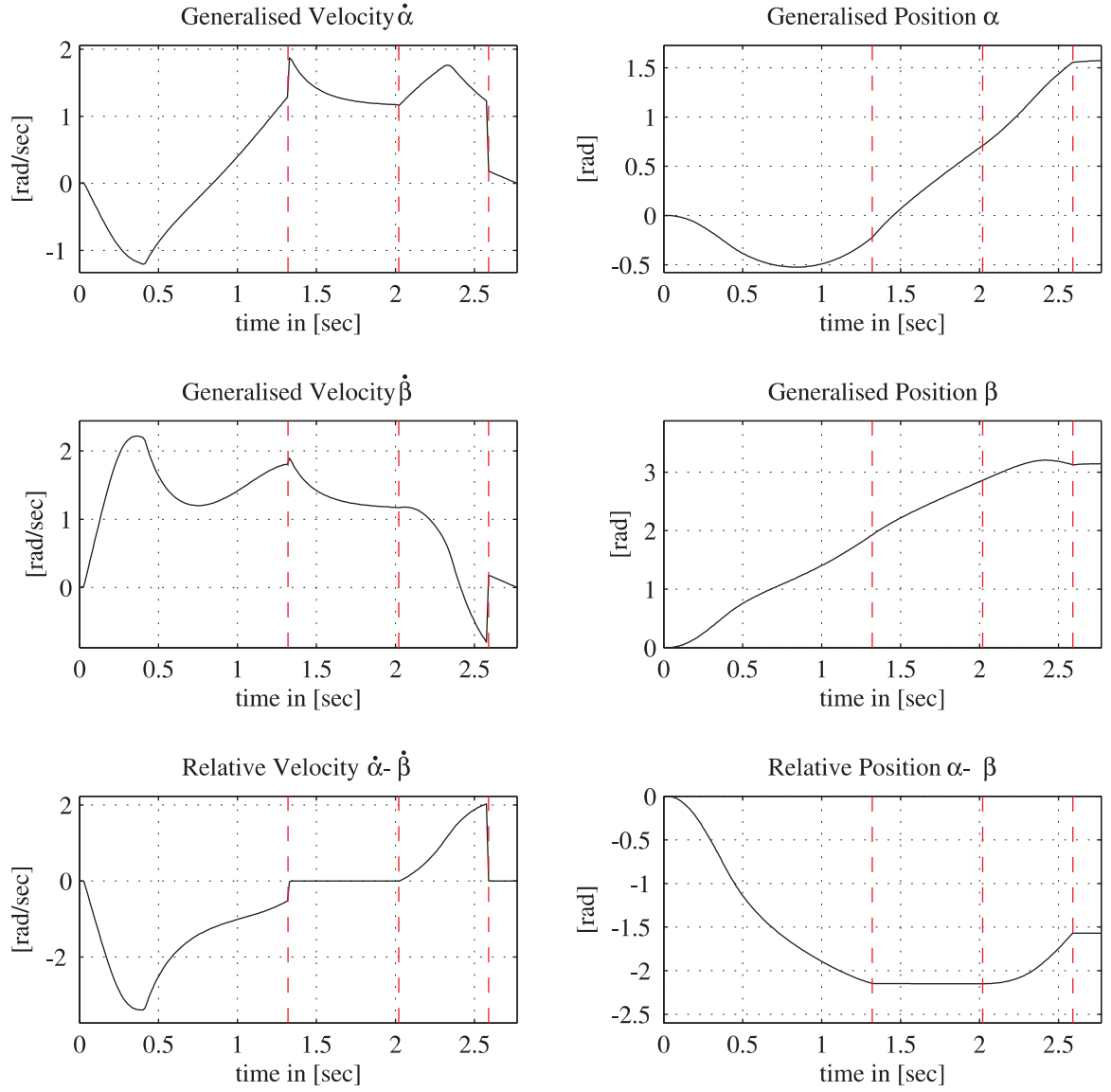


Figure 3.29: Case C: The optimal evolution of the generalised velocities $\dot{\alpha}$, $\dot{\beta}$, relative velocity $\dot{\alpha} - \dot{\beta}$, generalize positions α , β and relative position $\alpha - \beta$. red lines mark the transition times)

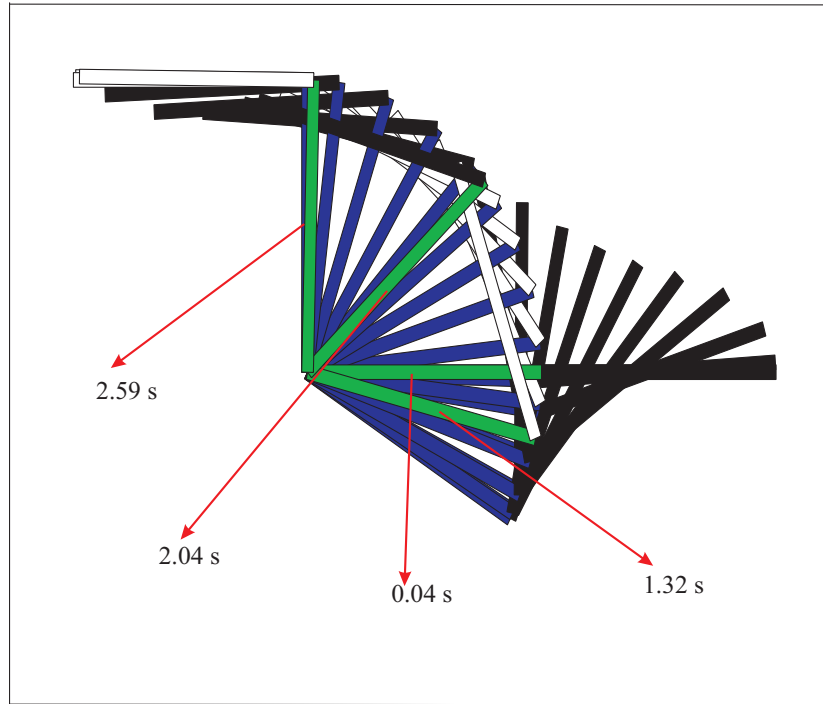


Figure 3.30: Case C: Maneuver of the double pendulum (Color code: For link 2 white is blocked, black is unblocked, for link 1 green position at transition interval).

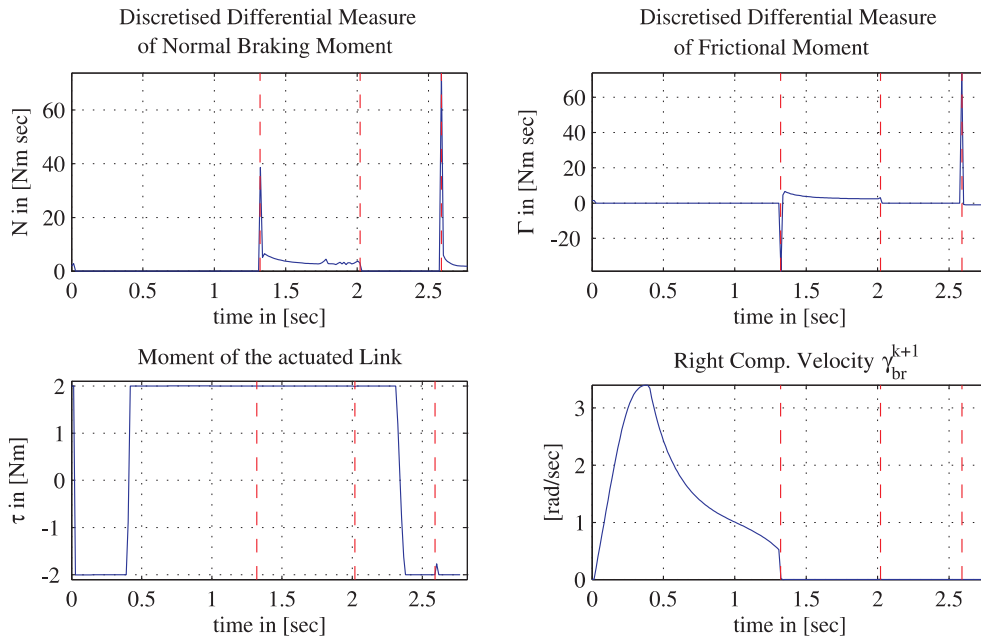


Figure 3.31: Case C: The optimal evolution of the discretised differential measures N , Γ , the torque τ and the slack velocity γ_{br} (red lines mark the transition times).

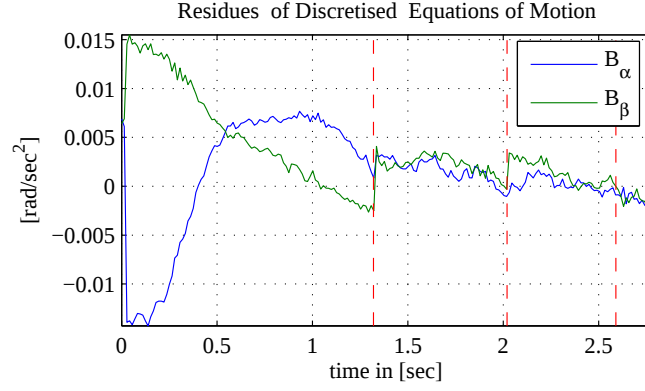


Figure 3.32: Case C: The numerical residue of the discretised equations of motion approx. by $\frac{\mathbf{B}[k]}{\Delta t}$ (red lines mark the transition times).

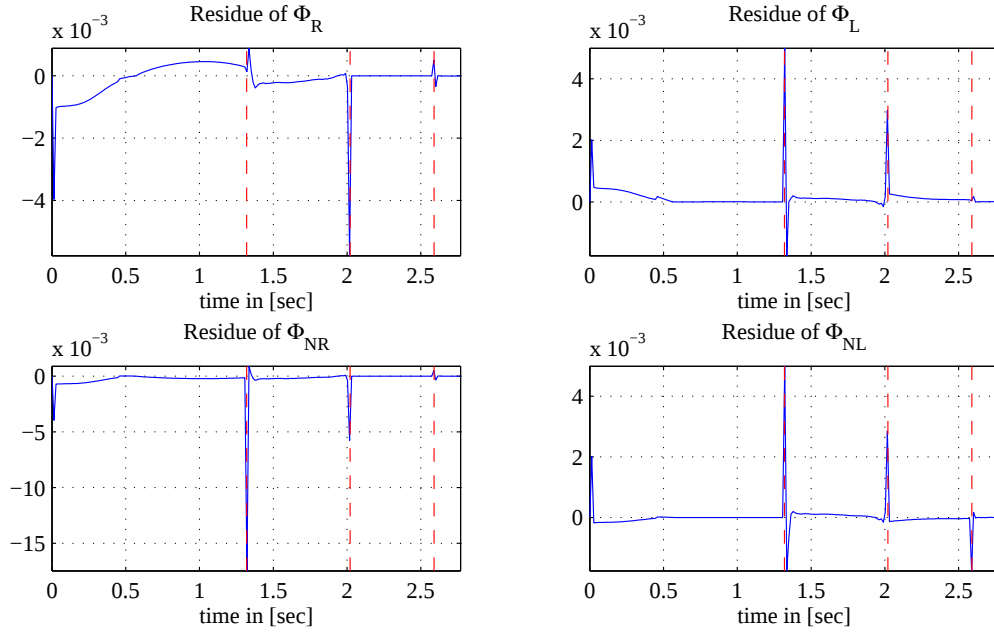


Figure 3.33: Case C: Residues of the reformulation functions Φ_R , Φ_L , Φ_{NR} and Φ_{NL} (red lines mark the transition times).

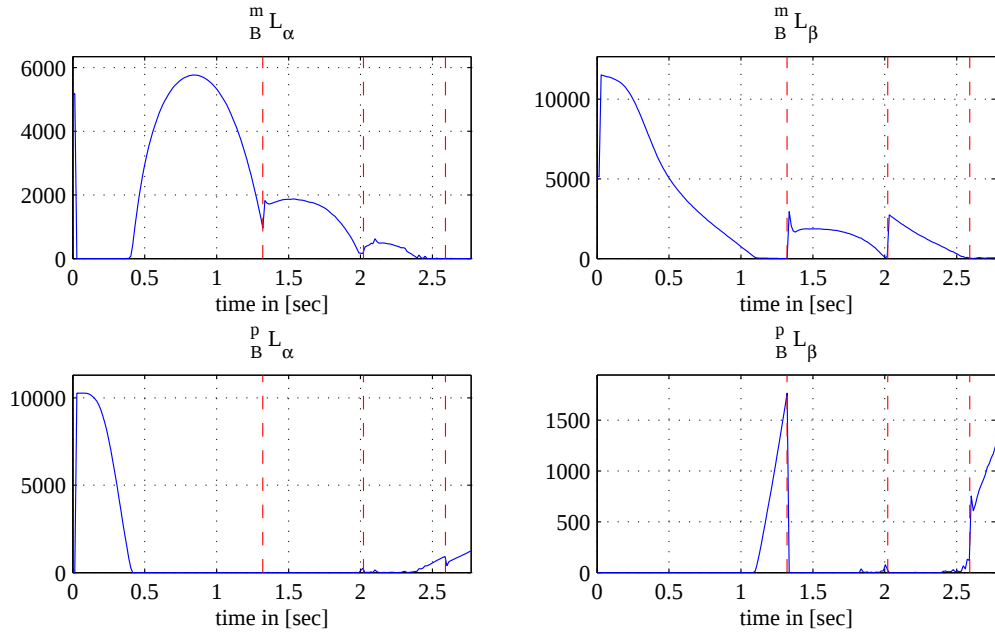


Figure 3.34: Case C: The optimal evolution of the dual multipliers ${}^m_B L_\alpha$, ${}^m_B L_\beta$, ${}^p_B L_\alpha$ and ${}^p_B L_\beta$ (red lines mark the transition times).

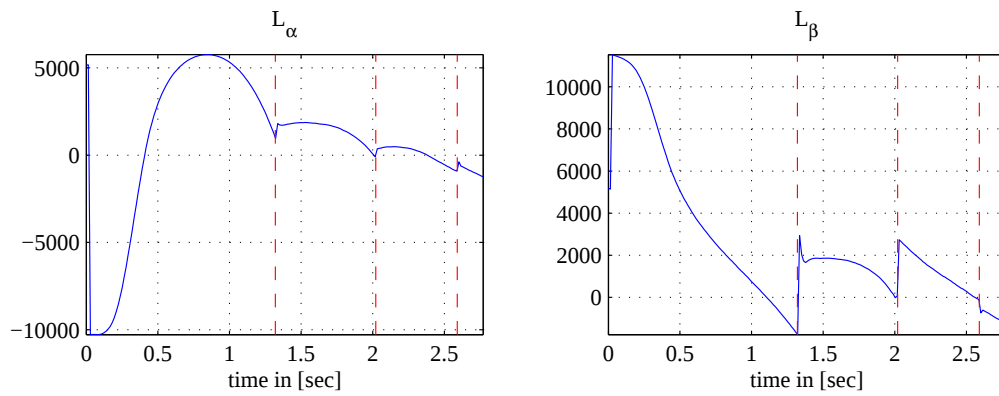


Figure 3.35: Case C: The time evolution of the costate dynamics in the second-order form (red lines mark the transition times).

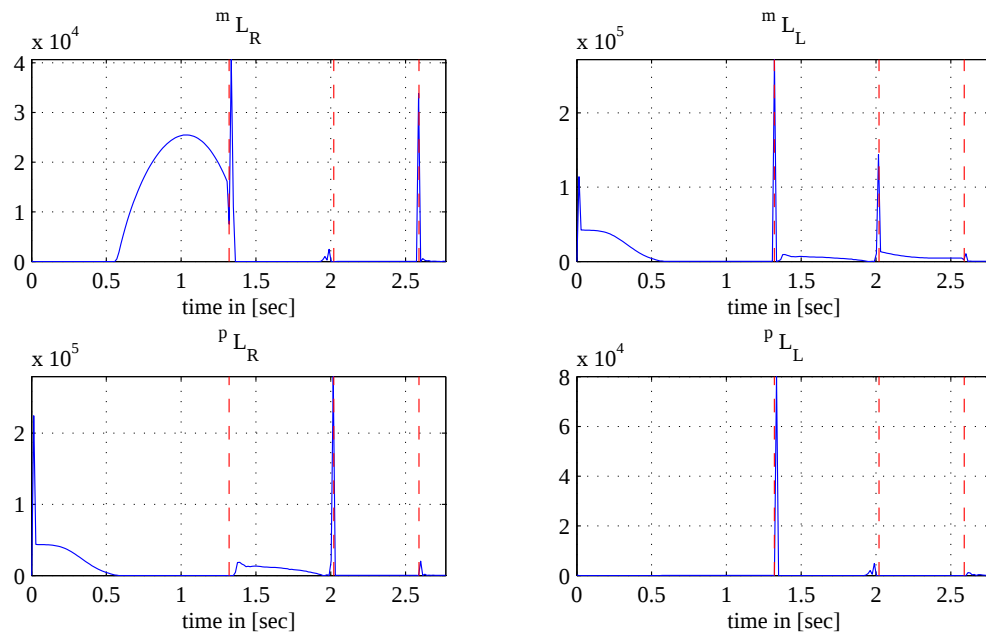


Figure 3.36: Case C: The optimal evolution of the dual multipliers ${}^m L_R$, ${}^m L_L$, ${}^p L_R$ and ${}^p L_L$ (red lines mark the transition times).

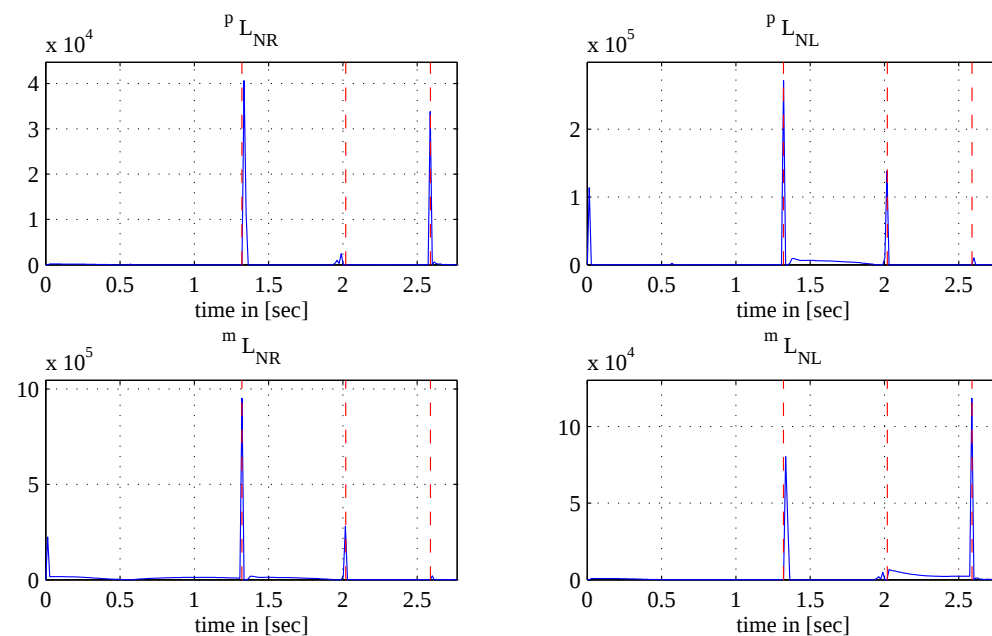


Figure 3.37: Case C: The optimal evolution of the dual multipliers ${}^m L_{NR}$, ${}^m L_{NL}$, ${}^p L_{NR}$ and ${}^p L_{NL}$ (red lines mark the transition times).

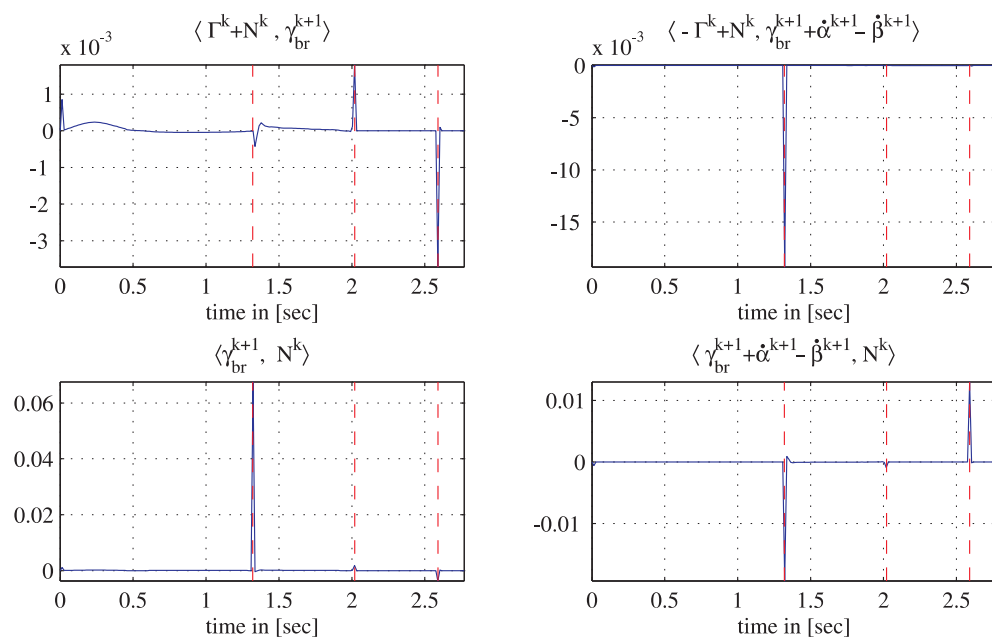


Figure 3.38: Case C: The residues of the complementarities (red lines mark the transition times).

3.5 An Augmented Lagrangian based Shooting Method

In recent years, several works have been presented in order to establish the relations between complementarity dynamical systems and hybrid systems. The treatment of discontinuous transitions and the combinatorial nature of mode sequencing are partially treated in these publications about optimal control, due to the modeling approach chosen. In this section, mechanical systems with autonomous transitions are considered, where the state trajectories remain continuous. For this class of mechanical systems, in [114] by Yunt *et al.* a time-stepping based shooting method is investigated for the first time, in [109] by Yunt the time-stepping based shooting is combined with the augmented Lagrangian method and the convergence proof is presented. In this work, the applicability of the shooting method is extended to the class of switching Lagrangian systems. In the sequel the definition of switching Lagrangian system is given and the concept of hybrid execution is introduced.

Definition Switching Lagrangian Systems A switching Lagrangian system is characterised by the following properties:

1. A set of discrete modes $\mathcal{I}_M$,
2. For each $l \in \mathcal{I}_M$, manifold $\mathcal{M}_l$, with boundary $\partial\mathcal{M}_l$, and a Lagrangian function $\mathcal{L}_l : \mathcal{T}\mathcal{M}_l \rightarrow \mathbb{R}$ defined on the tangent bundle of $\mathcal{M}_l$, where the domains $\mathcal{M}_l$ are disjoint.
3. A set of discrete transitions $\mathcal{E}_T \subset \mathcal{I}_M \times \mathcal{I}_M$;
4. For each discrete transition, there exists a set-valued force law that relates the acceleration of the Lagrangian system to the forces.
5. The contacts of the mechanical system on position level remain unchanged in their status, but directions of motion on velocity and acceleration level are closed and released.

The dynamical systems characterised by the definition of switching Lagrangian systems encompass the important subclass of nonholonomic mechanical systems.

Definition Hybrid Executions of switching Lagrangian Systems A hybrid execution also called a hybrid trajectory of a switching Lagrangian system is defined on a time interval $[t_0, t_f]$, if there exist a finite partition $\mathcal{P}$ of $[t_0, t_f]$,

$$\mathcal{P} = t_0 \leq t_1, \dots, t_{p+1} = t_f, \quad p \geq 0, \quad p \text{ integer} \quad (3.217)$$

and a succession of discrete modes $\{a_0, \dots, a_p\} \in \mathcal{I}_M$ and arcs $\pi_0, \dots, \pi_p$, such that

1. $(a_j, a_{j+1}) \in \mathcal{E}_T$ for $j = 0, \dots, p-1$;
2. $a_j : [t_j, t_{j+1}] \rightarrow \mathcal{M}_{l_j}$ is a continuous and piecewise smooth curve in $\mathcal{M}_{l_j}$ for $j = 0, \dots, p$.

There are many classes of switching hybrid dynamical systems and based on this diversity many optimal control approaches have been developed and investigated. The complexity of such problems poses challenges both theoretically and numerically. In two publications [106] and [107] Xu and Antsaklis present a direct numerical method for the switching time optimisation for systems without state discontinuity at transitions based on numerical and analytical differentiations of the value function. The issue of differentiability and continuity of the value function is a complex one, and requires the pre-specification of the mode sequence and partitioning of the considered hybrid optimal control problem in advance. In this work a shooting method is proposed, that determines the partition and mode sequence in addition to the controls and does not require any gradient information for the class of switching Lagrangian systems. Shaikh *et al.* present in [93] and [94] necessary and sufficient conditions for hybrid dynamical systems with state-continuous transitions. There are some works devoted to the complementarity modeling and optimal control of mechanical systems as given in references [109], [114], [115], [116]. In [67] T. Murphey addresses some aspects of non-smooth dynamics of mechanical systems, and control aspects from view point of stability are treated. There he introduces the class of *multiple-model systems*, referring to the fact that through contact interactions such as stick-slip transitions, the mechanical system can be represented by different differential equations depending on the state of the contacts. His approach falls generally into the approach of discrete-event system analysis in hybrid system terminology. The optimisation is performed by the direct shooting method combined with successive quadratic penalty method. The numerical integrations are performed by applying Moreau's time-stepping discretisation scheme to the measure-differential inclusion describing the mechanical dynamics and is rooted in his works such as [64], [65], [66]. Treating second-order mechanical processes in the framework measure-differential inclusions leads to an index reduction in comparison to differential equation representation with algebraic equality constraints. Because of this property, in the proposed optimisation method, the shooting scheme is not partitioned a priori by assuming a certain hybrid execution structure, but can be treated as a single interval of integration. The partition and sequence of modes is an outcome of the optimisation method. The contact dynamics arising from a spatial Coulomb friction are handled by a so-called exact-regularisation technique that has its roots in the augmented Lagrangian formulation of the Lagrangian dynamics as first described in [6]. Moreau's time-stepping discretisation scheme along with the exact-regularisation is expounded. The implementation of the optimisation with the Nelder-Mead simplex method and the direct shooting technique is elucidated before presenting results. The time-stepping integration method is adapted for the purposes of the shooting method because the shooting method requires several thousand hybrid executions in order to obtain a minimum.

- One of the main advantages of direct-search methods is that the minimisation is performed

by function value comparison, so that the problems of ill-conditioning due to high penalty parameters that degrade the performance of methods that use first and second-order information, is not a problem for such methods.

- Direct search methods have the disadvantage that their convergence in the vicinity of the solution is slow. A partial remedy to this problem in the framework of the augmented Lagrangian technique is the partial/incomplete minimisation approach in the intermediate stages of the successive minimisations.
- The local convexification of the feasible region, that is induced by the penalisation, guarantees that the unconstrained function is bounded-below and that through direct-search methods the some feasible region is reached. Moreover, first-order methods can get stuck in locally minimal solutions depending on severity of the non-convexity of the value function and direct-search methods are less prone to such a problem.

3.6 Treatment of Contact Dynamics and Time-Stepping Integration

The sweeping discretisation method has been since its introduction in the late sixties been investigated intensively as an integration routine. The existence of contact forces in the distributional sense, implies that the transitions indeed happen on a time-instant of Lebesgue measure zero if one considers the continuous description of the problem, on the other side in the discretisation it means that the transition happens on a discrete time interval on which the discretized differential measures of contact forces exist in the distributional sense. The continuous description is compliant to various hybrid system modeling approaches where transitions happen on a time instant of Lebesgue-measure zero, whereas the discretized treatment the problem enables the transition to take place on a time interval of length of the discretisation length. The differential measures on the other hand can be related to impulses as discussed in [112]. The mathematical framework for impulses is to consider them as dirac distributions. There are two main aspects in considering contact forces as impulses:

- Impulses are associated to jumps on velocity level, and contact force laws and rules on acceleration level become obsolete. This is the main mechanism behind the index set reduction.
- Dirac distributions provide a suitable mathematical modeling framework for impulses. So contact impulses exist then on intervals in a distributional sense.

Numerically, the replacement of the set-valued inclusions by equalities is the key issue in formulating the iterative scheme to calculate contact forces. The set-valued force laws are

rewritten into equalities by making use of the relation stated in equation (3.218):

$$\boldsymbol{\lambda} = \text{proj}_{\mathcal{C}}(\boldsymbol{\lambda} + \epsilon \mathbf{s}), \epsilon > 0, \quad \Leftrightarrow \boldsymbol{\lambda} \in \mathcal{N}_{\mathcal{C}}^*(\mathbf{s}) \quad \Leftrightarrow \mathbf{s} \in \mathcal{N}_{\mathcal{C}}(\boldsymbol{\lambda}). \quad (3.218)$$

Here $\mathbf{y} = \text{proj}_{\mathcal{C}}(\mathbf{x})$ denotes the nearest point $\mathbf{y} \in \mathcal{C}$ to $\mathbf{x}$, and $\mathcal{N}_{\mathcal{C}}^*$ is equivalent to the sub-differential of the support function of the convex set $\mathcal{C}$ in the sense of convex analysis. The real-valued parameter ϵ can take any positive value. A good exposition of the properties of projections to closed sets in the more general non-convex setting can be found in [31] by Clarke *et al.*. For switching Lagrangian systems where normal contact forces are passive forces in all modes, the normal forces can be determined by the projection of the change of linear and angular momenta in the constrained directions. Passive forces are characterized by their diminishing virtual power resulting from admissible virtual velocities which are compliant with the constraints. This projection enables to describe the evolution of the dynamics using less degrees of freedom, which reduces the numerical effort in the shooting procedure. The projection enables to obtain an algebraic equality for the normal contact forces in the following form:

$$\boldsymbol{\lambda}_u = \mathbf{P}_a(\mathbf{q}, \mathbf{u}, \dot{\mathbf{u}}, \boldsymbol{\tau}, \boldsymbol{\lambda}_s) = \mathbf{V}_a(\mathbf{q}, \mathbf{u}, \dot{\mathbf{u}}) + \mathbf{R}_a(\mathbf{q}) \boldsymbol{\lambda}_s + \mathbf{S}_a(\mathbf{q}) \boldsymbol{\tau}, \quad (3.219)$$

where $\mathbf{P}_a$ is a set of algebraic equations which are explicitly solved for the normal contact forces, $\mathbf{S}_a(\mathbf{q})$ and $\mathbf{R}_a(\mathbf{q})$ are $m \times s$ and $m \times 2m$ dimensional linear operators respectively. Since no adhesive forces at contacts are allowed, the normal contact forces have to be restrained to $\mathbb{R}_0^+$. The differential inclusion described by relations (2.40) to (2.42) is then reformulated as:

$$\mathbf{M}(\mathbf{q}) \dot{\mathbf{u}} - \mathbf{h}(\mathbf{q}, \mathbf{u}) - \mathbf{D}_T(\mathbf{q}) \boldsymbol{\lambda}_T - \mathbf{B}(\mathbf{q}) \boldsymbol{\tau} = \mathbf{0}, \quad (3.220)$$

$$\boldsymbol{\lambda}_N = \text{proj}_{\mathbb{R}_0^+}(\boldsymbol{\lambda}_N + \epsilon \mathbf{P}_a(\mathbf{q}, \mathbf{u}, \dot{\mathbf{u}}, \boldsymbol{\tau}, \boldsymbol{\lambda}_T)), \quad \forall i \in \mathcal{I}_N, \quad (3.221)$$

$$\boldsymbol{\lambda}_{T_i} = \text{proj}_{\mathcal{C}_{T_i}}(\boldsymbol{\lambda}_{T_i} - \epsilon \dot{\boldsymbol{\gamma}}_{T_i}), \quad \forall i \in \mathcal{I}_N. \quad (3.222)$$

Analogously, by the projection of the differential measures of angular and linear momenta, equations of the form as given in equation (3.223) can be obtained:

$$d\boldsymbol{\Lambda}_u = \mathbf{P}_v(\mathbf{q}, \mathbf{u}^+, d\mathbf{u}, \boldsymbol{\tau}, d\boldsymbol{\Lambda}_s) = \mathbf{V}_v(\mathbf{q}, \mathbf{u}^+, d\mathbf{u}) + \mathbf{R}_v(\mathbf{q}) d\boldsymbol{\Lambda}_s + \mathbf{S}_v(\mathbf{q}) \boldsymbol{\tau} dt, \quad (3.223)$$

where $\mathbf{P}_v$ is a set of algebraic equations which are explicitly solved for the normal contact forces, where $\mathbf{S}_v(\mathbf{q})$ and $\mathbf{R}_v(\mathbf{q})$ are $k \times s$ and $k \times 2k$ dimensional linear operators respectively. Since no adhesive forces at contacts are allowed, the differential measures of the normal contact forces have to be restrained to $\mathbb{R}_0^+$. Similarly, the equations (2.46), (2.47), (2.48) convert into:

$$\mathbf{M} d\mathbf{u} - \mathbf{W}_s(\mathbf{q}) d\boldsymbol{\Lambda}_s - (\mathbf{h}(\mathbf{q}, \mathbf{u}^+) + \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}) dt = \mathbf{0}, \quad (3.224)$$

$$d\boldsymbol{\Lambda}_u \in \mathcal{N}_{\mathbb{R}_0^+}(\mathbf{P}_v(\mathbf{q}, \mathbf{u}^+, d\mathbf{u}, \boldsymbol{\tau}, d\boldsymbol{\Lambda}_s)), \quad \forall i \in \mathcal{I}_S, \quad (3.225)$$

$$-\boldsymbol{\gamma}_{s_i}^+ \in \mathcal{N}_{\mathcal{C}_{s_i}(d\boldsymbol{\Lambda}_{n_i})}(d\boldsymbol{\Lambda}_{s_i}), \quad \forall i \in \mathcal{I}_S. \quad (3.226)$$

The above relation stated in (3.218) enables the reformulation of the set-valued relations from (3.224) to (3.226) in a system of equalities:

$$\mathbf{M} d\mathbf{u} - \mathbf{h}(\mathbf{q}, \mathbf{u}) dt - \mathbf{W}_s(\mathbf{q}) d\mathbf{\Lambda}_s - \mathbf{B}(\mathbf{q}) \boldsymbol{\tau} dt = \mathbf{0}, \quad (3.227)$$

$$d\mathbf{\Lambda}_u = \text{proj}_{\mathbb{R}_0^+} (d\mathbf{\Lambda}_u + \epsilon \mathbf{P}_v(\mathbf{q}, \mathbf{u}^+, d\mathbf{u}, \boldsymbol{\tau}, d\mathbf{\Lambda}_s)), \quad \forall i \in \mathcal{I}_S \quad (3.228)$$

$$d\mathbf{\Lambda}_{s_i} = \text{proj}_{\mathcal{C}_{s_i}(d\mathbf{\Lambda}_{u_i})} (d\mathbf{\Lambda}_{s_i} - \epsilon \gamma_{s_i}^+), \quad \forall i \in \mathcal{I}_S. \quad (3.229)$$

To perform numerical integration of a system with respect to time, one has to address the following problem; for given initial time t^j and known initial displacements $\mathbf{q}(t^j) = \mathbf{q}^j \in \mathbb{R}^n$ and velocities $\mathbf{u}^j = \mathbf{u}(t^j) \in \mathbb{R}^n$, find approximations of the displacements $\mathbf{q}(t^{j+1}) = \mathbf{q}^{j+1} \in \mathbb{R}^n$ and velocities $\mathbf{u}^{j+1} = \mathbf{u}(t^{j+1}) \in \mathbb{R}^n$ at the end t^{j+1} of a chosen time interval $[t^j, t^{j+1}]$. The procedure to obtain these approximations in the sense of the sweeping process is described by the following steps:

1. Given a time step $t^j + \Delta t = t^{j+1}$, compute the midpoint $t^m = t^j + \frac{1}{2} \Delta t$ and the endpoint $t^{j+1} = t^j + \Delta t$ of the time interval.
2. Approximate midpoint displacements by $\mathbf{q}^m = \mathbf{q}^j + \frac{1}{2} \Delta t \mathbf{u}^j \in \mathbb{R}^n$
3. Matrix calculations:
 - Compute $\mathbf{M}(\mathbf{q}^m) \in \mathbb{R}^{n \times n}$, $\mathbf{h}(\mathbf{q}^m, \mathbf{u}^j) \in \mathbb{R}^n$ and $\mathbf{B}(\mathbf{q}^m) \in \mathbb{R}^{n \times s}$
 - For $i = 1, \dots, k$ set up the index set $\mathcal{I}_S = \{i \mid g_{u_i}(\mathbf{q}^m) \leq 0\}$.
 - For every $i \in \mathcal{I}_S$ compute $\mathbf{w}_{s_i}(\mathbf{q}^m) \in \mathbb{R}^{n \times 2}$.
4. Determination of $\mathbf{u}^{j+1}$: In this step following equations have to be solved:

$$\mathbf{M} (\mathbf{u}^{j+1} - \mathbf{u}^j) - \mathbf{h} \Delta t - \sum_{\forall i \in \mathcal{I}_S} \mathbf{w}_{s_i} \mathbf{\Lambda}_{s_i} - \mathbf{B} \boldsymbol{\tau}^j \Delta t = \mathbf{0}, \quad (3.230)$$

$$\gamma_{s_i}^{j+1} = \mathbf{w}_{s_i}^T \mathbf{u}^{j+1}, \quad (3.231)$$

$$\mathbf{\Lambda}_u = \text{proj}_{\mathbb{R}_0^+} (\mathbf{\Lambda}_u + \epsilon \mathbf{P}_v(\mathbf{q}^m, \mathbf{u}^{j+1} - \mathbf{u}^j, \boldsymbol{\tau}^j, \mathbf{\Lambda}_s, \Delta t)), \quad (3.232)$$

$$\forall i \in \mathcal{I}_S,$$

$$\mathbf{\Lambda}_{s_i} = \text{proj}_{\mathcal{C}_{s_i}(\mathbf{\Lambda}_{u_i})} (\mathbf{\Lambda}_{s_i} - \epsilon \gamma_{s_i}^{j+1}), \quad \forall i \in \mathcal{I}_S. \quad (3.233)$$

5. Computation of $\mathbf{q}^{j+1} = \mathbf{q}^m + \frac{1}{2} \Delta t \cdot \mathbf{u}^{j+1}$.

The vectors $\mathbf{\Lambda}_s$ and $\mathbf{\Lambda}_u$ represent the discretised forms of the tangential and normal contact differential measures, that exist in a distributional sense on the interval $[t^j, t^{j+1}]$. The solution of step 4 is accomplished iteratively in the index k :

$$\text{Initialisation of Iteration} \quad \mathbf{\Lambda}_u^0 = \mathbf{0}, \quad \mathbf{\Lambda}_{s_i}^0 = \mathbf{0}, \quad \forall i \in \mathcal{I}_S. \quad (3.234)$$

$$\mathbf{u}_{k+1}^{j+1} = \mathbf{u}^j + \mathbf{M}^{-1}(\mathbf{q}^m) \left(\mathbf{h}(\mathbf{q}^m, \mathbf{u}^j) \Delta t + \sum_{\forall i \in \mathcal{I}_S} \mathbf{w}_{s_i}(\mathbf{q}^m) \boldsymbol{\Lambda}_{s_i}^k + \mathbf{B}(\mathbf{q}^m) \boldsymbol{\tau}^j \Delta t \right), \quad (3.235)$$

$${}_{k+1}\boldsymbol{\gamma}_{s_i}^{j+1} = \mathbf{w}_{s_i}^T \mathbf{u}_{k+1}^{j+1}, \quad \forall i \in \mathcal{I}_S, \quad (3.236)$$

$$\boldsymbol{\Lambda}_{s_i}^{k+1} = \text{proj}_{\mathcal{C}_{s_i}(\boldsymbol{\Lambda}_{u_i}^k)} (\boldsymbol{\Lambda}_{s_i}^k - \epsilon_{k+1} \boldsymbol{\gamma}_{s_i}^{j+1}), \quad \forall i \in \mathcal{I}_S, \quad (3.237)$$

$$\boldsymbol{\Lambda}_u^{k+1} = \text{proj}_{\mathbb{R}_0^+} \left(\boldsymbol{\Lambda}_u^k + \epsilon \mathbf{P}_v \left(\mathbf{q}^m, \mathbf{u}_{k+1}^{j+1} - \mathbf{u}^j, \boldsymbol{\tau}^j, \boldsymbol{\Lambda}_s^{k+1}, \Delta t \right) \right), \quad \forall i \in \mathcal{I}_S. \quad (3.238)$$

Here the set $\mathcal{C}_{s_i}$ is defined as:

$$\mathcal{C}_{s_i}(\boldsymbol{\Lambda}_{u_i}^k) = \{ \boldsymbol{\Lambda}_{s_i}^k \mid |\boldsymbol{\Lambda}_{s_i}^k| \leq \mu_i \boldsymbol{\Lambda}_{u_i}^k, \quad \forall i \in \mathcal{I}_S \}, \quad (3.239)$$

At each integration time point, the embedded iterations are stopped when the norm of change of successive discretised normal and tangential contact force measures

$$\sum_{\forall i \in \mathcal{I}_S} \|\boldsymbol{\Lambda}_{s_i}^{k+1} - \boldsymbol{\Lambda}_{s_i}^k\| + \|\boldsymbol{\Lambda}_{u_i}^{k+1} - \boldsymbol{\Lambda}_{u_i}^k\| \quad (3.240)$$

is less than a specified tolerance Tot_{iter} . More advanced discretisation schemes may be found in the literature, such as the powerful Θ -method, an algorithm based on displacements with proven convergence. The literature on the time-stepping based simulation of systems with friction is vast and good overviews are provided in [7], [97], [98], [99], [100]. The sweeping process is extended in [1] to higher-order dynamical systems.

3.6.1 Formulation of the Trajectory Optimisation of Mechanical Systems

The aim of the optimisation method is to obtain the optimal hybrid execution for the hybrid process in addition to the optimal controls as a function of time on a given partition $\mathcal{P}$. Given a switching finite-dimensional Lagrangian system described by the relations (3.224) to (3.226) with n general coordinates, the controls are discretized by N_c and the states and velocities are discretized by N_s points. The time-stepping integration defines a nonlinear discrete single-valued mapping $\mathcal{I} : \mathbb{R}^{2N+sN_c+1} \rightarrow \mathbb{R}^{2n}$, that relates the final position $\mathbf{q}_f$ and velocity $\mathbf{u}_f$ to the initial position $\mathbf{q}_0$, initial velocity $\mathbf{u}_0$ and the primal variables of optimisation $\mathbf{y}$ as follows:

$$\begin{bmatrix} \mathbf{q}_f \\ \mathbf{u}_f \end{bmatrix} = \mathcal{I}(\mathbf{y}, \mathbf{q}_0, \mathbf{u}_0). \quad (3.241)$$

Here $\mathbf{y}$ is composed of the actuating torques/forces and the final time $\mathbf{y} = [\boldsymbol{\tau}, t_f]^T \in \mathbb{R}^{sN_c+1}$. Furthermore let the final desired state be specified by a set of equalities/inequalities and constraints be imposed on the controls as well as the final time as follows:

$$\mathcal{F} = \left\{ \mathbf{y} \mid \begin{bmatrix} \boldsymbol{\tau}_{\min} \\ t_{f\min} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{\tau} \\ t_f \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{\tau}_{\max} \\ t_{f\max} \end{bmatrix} \right\}, \quad (3.242)$$

and let the fulfillment of the following set of equalities suffice for the reaching of the final state:

$$\Delta_f = \begin{bmatrix} \mathbf{q}_f - \mathbf{q}_d \\ \mathbf{u}_f - \mathbf{u}_d \end{bmatrix} = \mathbf{0} \in \mathbb{R}^{2n}. \quad (3.243)$$

Here $\mathbf{q}_d$ and $\mathbf{u}_d$ represent the desired end position and velocity of the mechanical system, respectively. Then the successive minimisation of the following augmented Lagrangian functional yields an at least locally optimal trajectory if a feasible set exists:

$$\begin{aligned} & \min_{\mathbf{y}} L_a(\mathbf{y}, \boldsymbol{\nu}^k, \boldsymbol{\mu}^k) \\ &= f(\mathbf{y}) + \langle \boldsymbol{\mu}^k, \Delta_f \rangle + \frac{c^k}{2} \langle \Delta_f, \Delta_f \rangle + \frac{1}{2c^k} (\langle \boldsymbol{\nu}^{k+1}, \boldsymbol{\nu}^{k+1} \rangle - \langle \boldsymbol{\nu}^k, \boldsymbol{\nu}^k \rangle) \end{aligned} \quad (3.244)$$

where $\boldsymbol{\nu}^{k+1}$ and $\boldsymbol{\mu}^{k+1}$ are given by:

$$\boldsymbol{\nu}^{k+1} = \text{proj}_{\mathbb{R}_0^+} (\boldsymbol{\nu}^k + c^k \mathbf{g}(\mathbf{y}^k)), \quad \boldsymbol{\mu}^{k+1} = \boldsymbol{\mu}^k + c^k \mathbf{h}(\mathbf{y}^k). \quad (3.245)$$

Here $\mathbf{y}^k$ is obtained as:

$$\mathbf{y}^k = \arg \min_{\mathbf{y}} L_a(\mathbf{y}, \boldsymbol{\nu}^k, \boldsymbol{\mu}^k). \quad (3.246)$$

The vectors $\boldsymbol{\mu} \in \mathbb{R}^{2n}$ and $\boldsymbol{\nu} \in \mathbb{R}^{2sN_c+2}$ denote the Lagrange multiplier vectors belonging to the equality and inequality constraint vectors, and $\mathbf{g}(\mathbf{y})$ the inequalities imposed on the controls and end time defined by the set $\mathcal{F}$. The successive minimisations of the augmented Lagrangian $L_a(\mathbf{y}, \boldsymbol{\nu}^k, \boldsymbol{\mu}^k)$ as $c_k \rightarrow +\infty$ and $c_{k+1} > c_k$ is assured to reveal a global or a local minimum if a nonempty solution set exists. The Nelder-Mead simplex method is used to perform the successive minimisations of $L_a(\mathbf{y}, \boldsymbol{\nu}^k, \boldsymbol{\mu}^k)$ with respect to $\mathbf{y}$, which is a non-gradient based minimisation algorithm. The Nelder-Mead method can be used to minimize a scalar-valued nonlinear function of w real variables using only function values, without any derivative information. The Nelder-Mead algorithm is a direct search algorithm, that maintains at each step a nondegenerate simplex, which is a geometric object in a w -dimensional space and is generated as the convex hull of $w + 1$ vertices. Since its introduction by Nelder and Mead in [71], the method belongs in the repertoire of engineering optimisation methods to the class of most frequently used methods, so it is astonishing enough to see that there are few works published investigating the properties of the method such as [50] and [56].

3.7 The Mechanical Modeling of the Differential Drive Robot

The proposed optimisation algorithm is applied to an example mechanical system. This system is a differential-drive robot. Nonholonomic mechanical systems such as wheeled robotic devices are used regularly in literature as benchmark systems for trajectory optimisation such as in [9] by Balkcom *et al.* and [69] by Murray *et al.*. In most of the publications the wheeled robots are treated as smooth mechanical systems that always fulfill the nonholonomic and rolling constraints. These models are either kinematic but seldomly dynamic. The proposed optimisation algorithm is applied to an example mechanical system. This system is a differential-drive robot. Its non-smooth dynamic model is capable of fully capturing the effects of structure-variance emanating from the planar stick-slip transitions at the wheel contacts. The model has first been introduced in [114] by Yunt *et al.* and in [109] by Yunt the convergence proof and the optimality certificate for the optimal trajectories are given. The differential-drive robot is a three-wheeled actuated robot of which the rear wheels are actuated and controlled separately contrary to the front wheel which is neither actuated nor steered. The differential-drive robot presented in this work has the ability to undergo stick-slip transitions at the wheel contacts. Therefore a dynamic model of a three-wheeled robot has been used. This non-smooth dynamic model is capable of fully capturing the effects of structure-variance emanating from the planar stick-slip transitions at the wheel contacts. There are several assumptions for the modeling of the system, in order to keep the complexity of the model adequate:

1. A rigid-body mechanical model is used.
2. The friction between wheels and ground is modeled as isotropic spatial Coulomb friction.
3. The non-steered unactuated front wheel is replaced by a stick as a simplification, removing two degrees of freedom (DOF) to be modeled, in order to fasten the shooting process.
4. The rotational inertia of the total actuation consisting of the components of motor rotors and transmissions are added to the rotational inertias of the wheels.
5. The separation of wheel contact from the floor is excluded but detected by making use of the nonnegativity of the normal contact forces. It is assumed that all wheels remain in contact with the floor.

Given these assumptions the differential-drive robot is parameterised by five degrees of freedom as shown in figure 3.40:

$$\mathbf{q} = \begin{bmatrix} x & y & \phi & \Psi_L & \Psi_R \end{bmatrix}^T. \quad (3.247)$$

$\mathbf{r}_{OA}$	vector pointing from O to A in the Euclidean space
$^S\mathbf{r}_{OA}$	representation of $\mathbf{r}_{OA}$ in frame S
$\mathbf{A}_{TS}$	coordinate transformation from the frame S to the frame T
$^U\boldsymbol{\Omega}_U, ^U\boldsymbol{\Psi}_U$	the angular velocity/acceleration of frame U
$^P\mathbf{N}_{S_U}^U$	spin of a rigid body U with respect to its CM S_U in the P -frame
$^S\boldsymbol{\Theta}_A^U$	inertia tensor of body U with respect to A in frame S
$^I\mathbf{W}_R^T$	The velocity jacobian of point R in frame I

Table 3.9: Notation Convention.

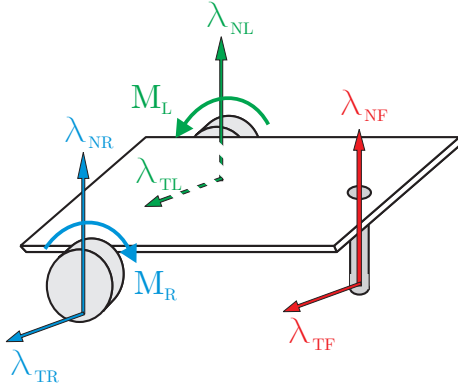


Figure 3.39: Contact forces and motor moments on the simplified model.

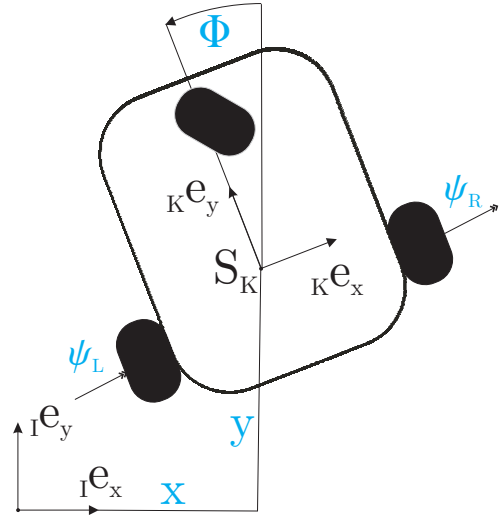


Figure 3.40: The degrees of freedom and coordinate frames of the wheeled robot.

The resulting mass matrix of the differential-drive robot has the following sparse structure:

$$\mathbf{M} = \begin{bmatrix} m_{11} & 0 & m_{13} & 0 & 0 \\ 0 & m_{22} & m_{23} & 0 & 0 \\ m_{31} & m_{32} & m_{33} & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 \\ 0 & 0 & 0 & 0 & m_{55} \end{bmatrix}. \quad (3.248)$$

The entries of the symmetric positive-definite mass matrix are given by:

$$\begin{aligned} m_{11} &= m_{22} = m_R + m_L + m_K, \\ m_{13} &= m_{31} = -m_L \sin(\phi)l_x - m_R \cos(\phi)r_y - m_R \sin(\phi)r_x - m_L \cos(\phi)l_y, \\ m_{33} &= 2b + k_{33} + m_R r_x^2 + m_R r_y^2 + m_L l_x^2 + m_L l_y^2, \\ m_{23} &= m_{32} = -m_L \sin(\phi)l_y + m_L \cos(\phi)l_x + m_R \cos(\phi)r_x - m_R \sin(\phi)r_y, \\ m_{44} &= m_{55} = a. \end{aligned}$$

The vector of gyroscopic and coriolis forces $\mathbf{h}$ does not contain any coupling between the actuated and nonactuated degrees of freedom. The vector $\mathbf{h}$ is expressed as:

$$\mathbf{h} = \begin{bmatrix} -\dot{\phi}^2 \mathbf{v} & \dot{\phi}^2 \mathbf{d} & 0 & 0 & 0 \end{bmatrix}^T. \quad (3.249)$$

The terms $\mathbf{v}$ and $\mathbf{d}$ are given by:

$$\begin{aligned} \mathbf{v} &= m_R \sin(\phi) \mathbf{r}_y - m_R \cos(\phi) \mathbf{r}_x + m_L \sin(\phi) \mathbf{l}_y - m_L \cos(\phi) \mathbf{l}_x, \\ \mathbf{d} &= m_L \cos(\phi) \mathbf{l}_y + m_L \sin(\phi) \mathbf{l}_x + m_R \cos(\phi) \mathbf{r}_y + m_R \sin(\phi) \mathbf{r}_x. \end{aligned}$$

The relative contact velocities at contact points C_R , C_L and C_F are denoted by γ_R , γ_L and γ_F , respectively. They are given by:

$${}^K \gamma_R = {}^K \mathbf{W}_{C_R}^T \mathbf{u}, \quad (3.250)$$

$${}^K \gamma_L = {}^K \mathbf{W}_{C_L}^T \mathbf{u}, \quad (3.251)$$

$${}^K \gamma_F = {}^K \mathbf{W}_{C_F}^T \mathbf{u}. \quad (3.252)$$

The linear operator of generalised friction force and generalised control force directions are as below:

$$\begin{aligned} {}^K \mathbf{W}_{C_R}^T &= \begin{bmatrix} \cos(\phi) & \sin(\phi) & -r_y & 0 & 0 \\ -\sin(\phi) & \cos(\phi) & r_x & 0 & r \end{bmatrix}, \\ {}^K \mathbf{W}_{C_L}^T &= \begin{bmatrix} \cos(\phi) & \sin(\phi) & -l_y & 0 & 0 \\ -\sin(\phi) & \cos(\phi) & l_x & r & 0 \end{bmatrix}, \\ \mathbf{W}_{C_F}^T &= \begin{bmatrix} 1 & 0 & -\sin(\phi)f_x - \cos(\phi)f_y & 0 & 0 \\ 0 & 1 & \cos(\phi)f_x - \sin(\phi)f_y & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \end{aligned}$$

respectively.

The positions of the center of masses (CM) of the chassis $\mathbf{S}_K$, and the relative positions of the right-wheel and left-wheel center of masses $\mathbf{S}_L$ and $\mathbf{S}_R$ with respect to $\mathbf{S}_K$ in the body-fixed coordinate frame K are given by:

$${}^I \mathbf{r}_{S_K} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}, \quad {}^K \mathbf{r}_{S_K S_R} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}, \quad {}^K \mathbf{r}_{S_K S_L} = \begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix}, \quad (3.253)$$

respectively. The center of mass velocities are :

Modes	${}^K\gamma_{Rx}$	${}^K\gamma_{Ry}$	${}^K\gamma_{Lx}$	${}^K\gamma_{Ly}$
5-DOF mode	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$
3R-DOF mode	$= 0$	$\neq 0$	$= 0$	$= 0$
3L-DOF mode	$= 0$	$= 0$	$= 0$	$\neq 0$
2-DOF mode	$= 0$	$= 0$	$= 0$	$= 0$

Table 3.10: The classification of modes by making use of the relative contact state of the rear wheels (The modes that emanate due to the stick-slide situations of the frontal contact C_F are neglected, $\mu_f \approx 0$)

$${}^I\mathbf{v}_{S_K} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix}, {}^I\mathbf{v}_{S_R} = \begin{bmatrix} \dot{x} - \cos(\phi)\dot{\phi}r_y - \sin(\phi)\dot{\phi}r_x \\ \dot{y} - \sin(\phi)\dot{\phi}r_y + \cos(\phi)\dot{\phi}r_x \\ 0 \end{bmatrix}, {}^I\mathbf{v}_{S_L} = \begin{bmatrix} \dot{x} - \cos(\phi)\dot{\phi}l_y - \sin(\phi)\dot{\phi}l_x \\ \dot{y} - \sin(\phi)\dot{\phi}l_y + \cos(\phi)\dot{\phi}l_x \\ 0 \end{bmatrix}$$

The relative positions of the stick contact point C_F , wheel contact points C_R and C_L with respect to S_K in the body fixed frame K are given by:

$${}^K\mathbf{r}_{S_K C_F} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, {}^K\mathbf{r}_{S_K C_R} = \begin{bmatrix} r_x \\ r_y \\ r_z - r \end{bmatrix}, {}^K\mathbf{r}_{S_K C_L} = \begin{bmatrix} l_x \\ l_y \\ l_z - r \end{bmatrix}, \quad (3.254)$$

respectively, where r denotes the wheel radius. The transformation matrices from the L - and R -Systems into the K -system are denoted as $\mathbf{A}_{KL}$ and $\mathbf{A}_{KR}$, respectively and are given by:

$$\mathbf{A}_{KR} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi_R) & -\sin(\psi_R) \\ 0 & \sin(\psi_R) & \cos(\psi_R) \end{bmatrix}, \mathbf{A}_{KL} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi_L) & -\sin(\psi_L) \\ 0 & \sin(\psi_L) & \cos(\psi_L) \end{bmatrix}.$$

The coordinate transformation from the chassis frame K to the inertial frame I is given by:

$$\mathbf{A}_{IK} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The inertia operators of each rigid-body is given as follows:

$${}^R\boldsymbol{\Theta}_{S_R}^R = {}^L\boldsymbol{\Theta}_{S_L}^L = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}, {}^K\boldsymbol{\Theta}_{S_K}^K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}. \quad (3.255)$$

The angular velocities of the rigid bodies with respect to their respective center of masses are

$${}_{\text{K}}\boldsymbol{\Omega}_{\text{K}} = {}_{\text{K}}\boldsymbol{\omega}_{\text{IK}} = \dot{\phi} \mathbf{e}_z^{\text{K}}, \quad {}_{\text{R}}\boldsymbol{\Omega}_{\text{R}} = \mathbf{A}_{\text{RK}} {}_{\text{K}}\boldsymbol{\omega}_{\text{IK}} + {}_{\text{R}}\boldsymbol{\omega}_{\text{KR}}, \quad {}_{\text{L}}\boldsymbol{\Omega}_{\text{L}} = \mathbf{A}_{\text{LK}} {}_{\text{K}}\boldsymbol{\omega}_{\text{IK}} + {}_{\text{L}}\boldsymbol{\omega}_{\text{KL}}.$$

The vectors of tangential and normal contact forces as well as controls are given by:

$${}_{\text{K}}\boldsymbol{\lambda}_{\text{T}} = \begin{bmatrix} \lambda_{\text{TRx}} \\ \lambda_{\text{TRY}} \\ \lambda_{\text{TLx}} \\ \lambda_{\text{TLy}} \\ \lambda_{\text{TFx}} \\ \lambda_{\text{TFy}} \end{bmatrix}, \quad {}_{\text{K}}\boldsymbol{\lambda}_{\text{N}} = \begin{bmatrix} \lambda_{\text{NR}} \\ \lambda_{\text{NL}} \\ \lambda_{\text{NF}} \end{bmatrix}, \quad \boldsymbol{\tau} = \begin{bmatrix} M_{\text{R}} \\ M_{\text{L}} \end{bmatrix}. \quad (3.256)$$

The Lagrangian system has four modes of operation, and their are classified according to the contact relative velocities in table 3.10. The set

$$\mathcal{I}_{\text{M}} = \{2\text{-DOF mode}, 3\text{R-DOF mode}, 3\text{L-DOF mode}, 5\text{-DOF mode}\}$$

includes all the four modes, the transitions among all four operating modes are possible. When the nonholonomic constraints and rolling constraints are fulfilled, the nonactuated Lagrangian system possesses two DOF and is fully actuated. In the two DOF the system possesses indeed three translational degrees of freedom but due to two constraints of nonholonomic character which are linearly dependent, the degrees of freedom on velocity level reduces to two.

If both wheels slide it is a Lagrangian system with five DOF with three underactuated DOF. In the three-DOF mode one wheel contact sticks and the other wheel slides, meaning that the nonholonomic constraints, due to wheel axis geometry, are fulfilled but one wheel does not fulfill the rolling condition. As again in the two-DOF mode, in the three-DOF modes the system possesses four translational degrees of freedom but due to one sticking wheel a nonholonomic constraint is present and the system possesses on velocity level actually only three degrees of freedom. If the actuation of the differential-drive robot is considered then one observes that the system is fully actuated when it moves in the two-DOF mode, in the other modes it is an underactuated system with less actuators than mechanical degrees of freedom. An interesting feature that this system exhibits regarding underactuation is the coupling between actuated and nonactuated degrees of freedom of the underactuated Lagrangian system. If the 5-DOF mode is taken as the basis then the underactuated DOF are the chassis DOF x, y, ϕ and the actuated DOF are ψ_{R} and ψ_{L} . The classes of underactuated systems investigated so far in literature, make use of the dynamical coupling in the vector $\mathbf{h}$, mass matrix $\mathbf{M}$ in order to affect the underactuated degrees of freedom. This coupling has a single-valued and nonlinear character. In the model of the differential-drive robot, however, the controlled DOF effect the nonsteered

DOF via contact forces which are set-valued. This means that in the absence of friction, the turning of the wheels can not move the robot, which can also be deduced by investigating the structure of the equations of motion. The vector $\mathbf{h}$ does not contain any coupling between the actuated and nonactuated degrees of freedom as given in (3.249). The sparse structures of the mass matrix $\mathbf{M}$ and the linear operator $\mathbf{B}$ indicate that the accelerations $\ddot{\Psi}_R$ and $\ddot{\Psi}_L$ are decoupled from the accelerations of the chassis translational and angular accelerations in the mass-matrix and that the control forces do not directly act on the accelerations $\ddot{x}$, $\ddot{y}$ and $\ddot{\phi}$. The structures of ${}_K\mathbf{W}_{C_R}^T$, ${}_K\mathbf{W}_{C_L}^T$ and $\mathbf{B}$ show that the underactuated chassis DOF x, y, ϕ are driven by the contact friction forces. The control torque is transferred, via set-valued coupling to the underactuated part of the dynamics. The nonuniqueness of the coupling arises due to contact geometry. In the 2-DOF mode during which the rear wheel contacts stick, the friction forces are nonunique and fulfill following set-valued relation:

$$({}_K\lambda_{TRx} + {}_K\lambda_{TLx})^2 + ({}_K\lambda_{TRy} + {}_K\lambda_{TLy})^2 \leq \mu^2 ({}_K\lambda_{NL} + {}_K\lambda_{NR})^2. \quad (3.257)$$

In the 3-DOF modes the friction forces of sticking wheel are unique, because due to sliding in one wheel contact the sliding forces can be located on the friction disk uniquely. In the 5-DOF mode the contact forces are unique.

The modes can be classified based on dissipative capacity, where the five-DOF mode has the highest dissipative capacity and the two-DOF mode, at which both wheel contacts stick is the at least dissipative mode. The two three-DOF modes have less dissipative capacity then the five-DOF mode.

3.7.1 Determination of Normal Contact Force Differential Measures

If the wheel contacts do not open during the course of motion and act as passive forces, then the normal forces can be regarded as bilateral constraint forces as long as they remain nonnegative. In order to determine the contact forces, the time rates of change of the angular and linear momenta of the whole rigid-body Lagrangian system has been determined with respect to the chassis CM of S_K and are denoted as ${}_K\dot{\mathbf{L}}_{S_K}^{\text{total}}$ and ${}_K\dot{\mathbf{p}}_{S_K}^{\text{total}}$. The projection of ${}_K\dot{\mathbf{p}}_{S_K}^{\text{total}}$ and ${}_K\dot{\mathbf{L}}_{S_K}^{\text{total}}$ in the constrained directions of motion which in this case arise from the impenetrability of the wheel contacts provides algebraic equations for the normal contact forces λ_{NF} , λ_{NR} and λ_{NL} . The normal contact forces are introduced to the model parallel to the $\mathbf{e}_z^I$ axis as shown in figure 4.1. If it is assumed that the wheel contacts do not open on position level then there are three DOF less necessary to model the robot, which reduces the size of equations of motions. By this approach the opening of a contact is detected but the evolution after opening of any contact is not evaluated because of lack of adequate parametrisation of the tangent space $\mathcal{T}_{\mathcal{M}}$. The

m_R	mass of right wheel [kg]	0.287	μ	friction coeff. at wheels	0.4
m_L	mass of left wheel [kg]	0.287	μ_f	friction coeff. at stick	0.01
m_K	mass of chassis [kg]	19.466	r	wheel radius [m]	0.0385
f_x	x-comp. of ${}_K\mathbf{r}_{S_K C_F}$ [m]	0.00876	f_y	y-comp. of ${}_K\mathbf{r}_{S_K C_F}$ [m]	0.1794
f_z	z-comp. of ${}_K\mathbf{r}_{S_K C_F}$ [m]	-0.0473	r_x	x-comp. of ${}_K\mathbf{r}_{S_K S_R}$ [m]	0.168
r_y	y-comp. of ${}_K\mathbf{r}_{S_K S_R}$ [m]	-9.67×10^{-2}	r_z	z-comp. of ${}_K\mathbf{r}_{S_K S_R}$ [m]	-0.0088
l_x	x-comp. of ${}_K\mathbf{r}_{S_K S_L}$ [m]	-0.1504	l_y	y-comp. of ${}_K\mathbf{r}_{S_K S_L}$ [m]	-9.67×10^{-2}
l_z	z-comp. of ${}_K\mathbf{r}_{S_K S_L}$ [m]	-0.0088	a	wheel inertia [kg m ²]	1.6778×10^{-4}
b	wheel inertia [kg m ²]	1.5604×10^{-4}	k_{11}	chassis inertia [kg m ²]	0.3646
k_{12}	chassis inertia [kg m ²]	0.0372	k_{13}	chassis inertia [kg m ²]	0.026
k_{21}	chassis inertia [kg m ²]	0.0372	k_{22}	chassis inertia [kg m ²]	0.2505
k_{23}	chassis inertia [kg m ²]	1.46×10^{-4}	k_{31}	chassis inertia [kg m ²]	0.026
k_{32}	chassis inertia [kg m ²]	1.46×10^{-4}	k_{33}	chassis inertia [kg m ²]	0.4306

Table 3.11: Numerical values of physical parameters of the robot. The numerical values are obtained from the CAD of a real mobile robot.

equations resulting from the projections are obtained as:

$$\langle {}_K(d\mathbf{p})_{S_K}^{\text{total}} - {}_K\mathbf{F}^{\text{total}} dt, \mathbf{e}_z^K \rangle = 0, \quad (3.258)$$

$$\langle {}_K(d\mathbf{L})_{S_K}^{\text{total}} - {}_K\mathbf{M}_{S_K}^{\text{total}} dt, \mathbf{e}_x^K \rangle = 0, \quad (3.259)$$

$$\langle {}_K(d\mathbf{L})_{S_K}^{\text{total}} - {}_K\mathbf{M}_{S_K}^{\text{total}} dt, \mathbf{e}_y^K \rangle = 0. \quad (3.260)$$

Here $d\mathbf{p}$ and $d\mathbf{L}$ denote the differential measure of the linear and angular momenta, respectively. ${}_K\mathbf{M}_{S_K}^{\text{Tot}}$ consists of all the moment contributions of the contact forces and weight with respect to S_K and results from the moment sum of all active forces and moments:

$$\begin{aligned} {}_K\mathbf{M}_{S_K}^{\text{Tot}} &= {}_K\mathbf{M}_{S_K}^K + {}_K\mathbf{M}_{S_K}^R + {}_K\mathbf{M}_{S_K}^L \\ &= {}_K\mathbf{r}_{S_K C_F} \times ({}_K\boldsymbol{\lambda}_{NF} + {}_K\boldsymbol{\lambda}_{TF}) + {}_K\mathbf{r}_{S_K C_R} \times ({}_K\boldsymbol{\lambda}_{NR} + {}_K\boldsymbol{\lambda}_{TR}) \\ &\quad + {}_K\mathbf{r}_{S_K C_L} \times ({}_K\boldsymbol{\lambda}_{NL} + {}_K\boldsymbol{\lambda}_{TL}) + {}_K\mathbf{r}_{S_K S_R} \times m_R g \mathbf{e}_z^K + {}_K\mathbf{r}_{S_K S_L} \times m_L g \mathbf{e}_z^K. \end{aligned}$$

The sum of external forces that act on the differential-drive robot is given by:

$${}_K\mathbf{F}^{\text{total}} = - \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} (m_L + m_R + m_K) + \begin{bmatrix} \lambda_{TRx} + \lambda_{TLx} + \lambda_{TFx} \\ \lambda_{TRY} + \lambda_{TLY} + \lambda_{TFy} \\ \lambda_{NR} + \lambda_{NL} + \lambda_{NF} \end{bmatrix}. \quad (3.261)$$

The operation $\times$ defines a cross product. The time rate of change of angular momentum

for each rigid body with respect to $\mathbf{S}_K$ is given by the expressions

$$\mathbf{M}_{S_K}^K = \dot{\mathbf{L}}_O^K + \mathbf{r}_{S_K O} \times \dot{\mathbf{p}}^K = \boldsymbol{\Theta}_{S_K}^K \boldsymbol{\Psi}_K + \boldsymbol{\Omega}_K \times \boldsymbol{\Theta}_{S_K}^K \boldsymbol{\Omega}_K, \quad (3.262)$$

$$\mathbf{M}_{S_K}^R = \dot{\mathbf{L}}_O^R + \mathbf{r}_{S_K O} \times \dot{\mathbf{p}}^R = \boldsymbol{\Theta}_{S_K}^R \boldsymbol{\Psi}_R + \boldsymbol{\Omega}_R \times \boldsymbol{\Theta}_{S_K}^R \boldsymbol{\Omega}_R + m_R \mathbf{r}_{S_K S_R} \times \mathbf{a}_{S_K}, \quad (3.263)$$

$$\mathbf{M}_{S_K}^L = \dot{\mathbf{L}}_O^L + \mathbf{r}_{S_K O} \times \dot{\mathbf{p}}^L = \boldsymbol{\Theta}_{S_K}^L \boldsymbol{\Psi}_L + \boldsymbol{\Omega}_L \times \boldsymbol{\Theta}_{S_K}^L \boldsymbol{\Omega}_L + m_L \mathbf{r}_{S_K S_L} \times \mathbf{a}_{S_K}. \quad (3.264)$$

The vector $\mathbf{a}_{S_K}$ is the translational acceleration of the chassis. The addition of expressions (3.262) to (3.264) yields:

$${}^T \dot{\mathbf{L}}_{S_K} = \dot{\mathbf{N}}_{S_K}^K + \dot{\mathbf{N}}_{S_R}^R + m_R \mathbf{r}_{S_K S_R} \times \mathbf{a}_{S_R} + \dot{\mathbf{N}}_{S_L}^L + m_L \mathbf{r}_{S_K S_L} \times \mathbf{a}_{S_L} = \mathbf{M}_{S_K}^{\text{Tot}}. \quad (3.265)$$

The time rate of change of the spin of a rigid-body U with respect to its center of mass in body fixed coordinates is given by:

$${}_U(\dot{\mathbf{N}}_U^U) = {}_U \boldsymbol{\Theta}_{S_U}^U \boldsymbol{\Psi}_U + {}_U \boldsymbol{\Omega}_U \times {}_U \boldsymbol{\Theta}_{S_U}^U \boldsymbol{\Omega}_U.$$

The differential measures of the total angular momentum of the rigid body system with respect to S_K is given by:

$$\begin{aligned} {}_K(d\mathbf{L})_{S_K}^{\text{total}} = \\ {}_K(d\mathbf{N})_{S_K}^K + {}_K(d\mathbf{N})_{S_R}^R + {}_K \mathbf{r}_{S_K S_R} \times m_R {}_K d\mathbf{v}_{S_R} + {}_K(d\mathbf{N})_{S_L}^L + {}_K \mathbf{r}_{S_K S_L} \times m_L {}_K d\mathbf{v}_{S_L} \end{aligned}$$

The differential measure of the spin is given by:

$${}_U(d\mathbf{N})_U^U = {}_U \boldsymbol{\Theta}_{S_U}^U (d\boldsymbol{\Omega})_U + {}_U \boldsymbol{\Omega}_U \times {}_U \boldsymbol{\Theta}_{S_U}^U \boldsymbol{\Omega}_U dt,$$

In the absence of impulsive forces the differential measures of linear and angular velocities are given by:

$$d\mathbf{v}_{S_\Theta} = \mathbf{a}_{S_\Theta} dt, \quad d\boldsymbol{\Omega}_{S_\Theta} = \boldsymbol{\Psi}_{S_\Theta} dt, \quad \forall \Theta \in \{L, R, K\}. \quad (3.266)$$

For the differential-drive robot the projections given in equations (3.258), (3.259) and (3.260) reveal corresponding operators in the projection (3.219) as given below:

$$\mathbf{V}_a = \mathbf{E}^{-1} \mathbf{F}, \quad \mathbf{R}_a = \mathbf{E}^{-1} \mathbf{G}^T \ddot{\mathbf{q}}, \quad \mathbf{S}_a = \mathbf{0}, \quad (3.267)$$

where the linear operators $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{G}$ are as given below:

$$\mathbf{E} = \begin{bmatrix} -r_y & -l_y & -f_y \\ r_x & l_x & f_x \\ -1 & -1 & -1 \end{bmatrix}, \quad (3.268)$$

$$\mathbf{F} = \begin{bmatrix} \dot{\phi}^2 k_{23} - r_z m_R \dot{\phi}^2 r_y - l_z m_L \dot{\phi}^2 l_y - r_y m_R g - l_y m_L g \\ -\dot{\phi}^2 k_{13} - \dot{\phi} \dot{\Psi}_R a + r_z m_R \dot{\phi}^2 r_x - \dot{\phi} \dot{\Psi}_L a + l_z m_L \dot{\phi}^2 l_x + r_x m_R g + l_x m_L g \\ -(m_K + m_R + m_L) g \end{bmatrix}, \quad (3.269)$$

$$\mathbf{G} = \begin{bmatrix} -\sin(\phi)r_z m_R - \sin(\phi)l_z m_L & -\cos(\phi)r_z m_R - \cos(\phi)l_z m_L & 0 \\ \cos(\phi)r_z m_R + \cos(\phi)l_z m_L & -\sin(\phi)r_z m_R - \sin(\phi)l_z m_L & 0 \\ -k_{13} + r_z m_R r_x + l_z m_L l_x & -k_{23} + r_z m_R r_y + l_z m_L l_y & 0 \\ -a & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}. \quad (3.270)$$

Further, if all three contacts belong to the index set $\mathcal{I}_S$, which is the case in the maneuvers of the robot based on the assumptions, then:

$$\mathbf{V}_a = \mathbf{V}_v, \quad \mathbf{R}_v = \mathbf{E}^{-1} \mathbf{G}^T d\mathbf{u}, \quad \mathbf{S}_v = \mathbf{0} \quad (3.271)$$

is valid.

3.8 Numerical Results

In this section four maneuvers are presented. The goal function $f(\mathbf{y})$ in (3.244) becomes

$$f(\mathbf{y}) = t_f \quad (3.272)$$

in the time-optimal case; or

$$f(\mathbf{y}) = \frac{t_f}{N_c} \sum_{i=1}^{N_c} M_L(i)^2 + M_R(i)^2 \quad (3.273)$$

in the control-effort case. The control moments of both wheels are limited to $\|M_L\| = 1 \text{ Nm}$ and $\|M_R\| = 1 \text{ Nm}$, respectively. A number of 300 discretisation points are used and the controls are discretized with 60 points each. The numerical values which are used in the optimisation are given in table 4.3 and 5.1. In all maneuvers the robot starts at the origin given by:

$$\begin{bmatrix} x_0, y_0, \phi_0, \psi_{L0}, \psi_{R0} \end{bmatrix} = \begin{bmatrix} 0, 0, 0, 0, 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x}_0, \dot{y}_0, \dot{\phi}_0, \dot{\psi}_{L0}, \dot{\psi}_{R0} \end{bmatrix} = \begin{bmatrix} 0, 0, 0, 0, 0 \end{bmatrix}.$$

In all the maneuvers, the robot stops at final time:

$$\begin{bmatrix} \dot{x}_f, \dot{y}_f, \dot{\phi}_f, \dot{\psi}_{Lf}, \dot{\psi}_{Rf} \end{bmatrix} = \begin{bmatrix} 0, 0, 0, 0, 0 \end{bmatrix}.$$

ϵ	0.1	t_{fmin}	2
c	10^3	t_{fmax}	5
c_d	10^6	M_{min}	-1
Tol_{iter}	10^{-7}	M_{max}	1
c_t	10^6	N_c	60
N_s	300		

Table 3.12: Numerical parameters of the optimisations in Maneuvers A, B, C and D. Here c_t and c_d are penalties imposed on constraints on end time and on the deviation from the final state, respectively.

3.8.1 Case A

The desired end position $\mathbf{q}_d$ to be reached time-optimally is given by:

$$\begin{bmatrix} x_f, y_f, \phi_f, \psi_{Lf}, \psi_{Rf} \end{bmatrix} = \begin{bmatrix} 1, 5, -\frac{\pi}{2}, \text{free}, \text{free} \end{bmatrix}.$$

Maneuver A is characterized by a high dynamical activity in the orientation of the chassis. The robot accomplishes this task in 4.03 seconds and the squared sum of the control effort is $3.73 \text{ N}^2 \text{ m}^2 \text{ s}$. In order to orient itself to $\phi = -\frac{\pi}{2}$ time-optimally in the final part of the maneuver the system performs a swing in maneuver during which it is in a five-DOF mode. At the beginning the system rushes onto trajectory in the vicinity $x = 1$ and swings counterclockwise out in order to have enough angular displacement to perform the swinging in the clockwise direction as can be seen in figure 3.43. Figure 3.41 shows the contact forces during the maneuver which are obtained by dividing the discretised differential measures of contact forces by the discretisation interval. The same figure shows the absolute values of the relative tangential contact velocities, which are used to analyse the mode behaviour of the system.

3.8.2 Case B

In maneuver B the task is to reach the following position $\mathbf{q}_d$ control-effort optimally:

$$\begin{bmatrix} x_f, y_f, \phi_f, \psi_{Lf}, \psi_{Rf} \end{bmatrix} = \begin{bmatrix} 1, 5, -\frac{\pi}{2}, \text{free}, \text{free} \end{bmatrix}.$$

The robot accomplishes this task in 4.21 seconds and the squared sum of the control effort is $3.57 \text{ N}^2 \text{ m}^2 \text{ s}$. As can be seen in figure 3.44 the CM of the robot traces a relatively straight line until the final reorientation maneuver during which dissipates kinetic energy in the process of brake down by the making use of the frictional work at the wheel contacts. The energy to the system is delivered by the control effort and in the case where the control effort is minimised, it

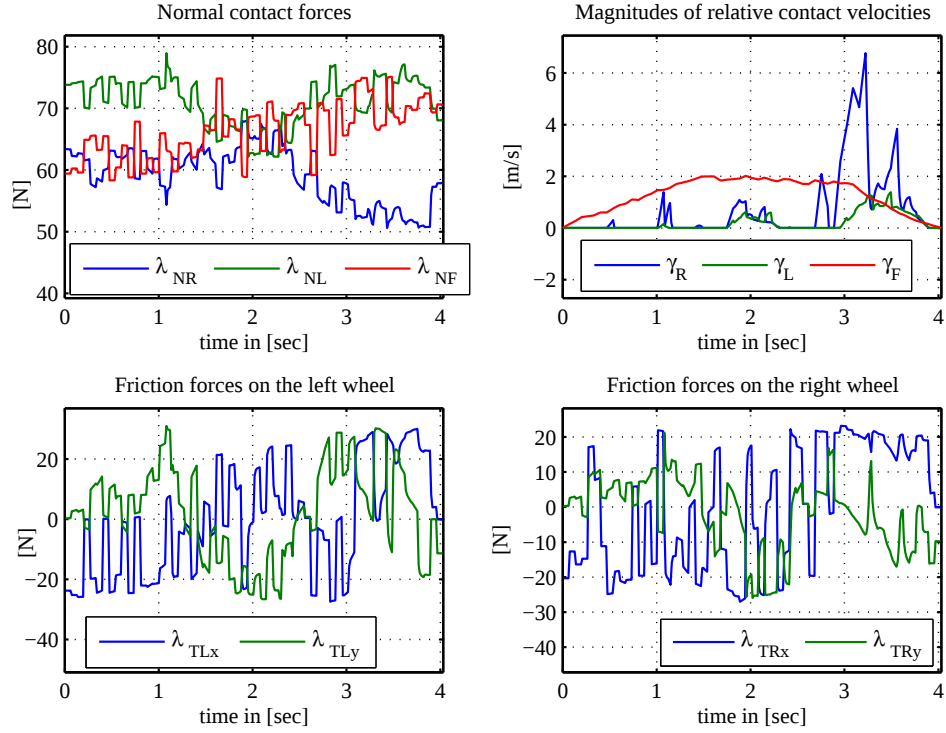


Figure 3.41: Case A: Contact forces and contact relative velocities.

is intuitively clear that the system tends not to dissipate energy at wheel contacts unnecessarily. As a consequence the Lagrangian system moves more in the non-dissipative two-DOF mode in the first half of the maneuver where the system is accelerated. Dissipation is preferred in order to brake down the system instead of making use of the control effort. In comparison to the time-optimal maneuver A, it moves less in dissipative modes which can also be seen in the trends of the magnitudes of the relative contact velocities of figures 3.41 and 3.42. The evolution of the center of mass trajectories in the x-y plane for maneuvers A and B can be seen in figures 3.47 and 3.48, respectively. The evolution of final time t_f over the successive minimisation of the augmented Lagrangian in both cases are depicted in figures 3.45 and 3.46, respectively.

3.8.3 Case C

In maneuver C the task is to reach the following position $\mathbf{q}_d$ time-optimally:

$$\begin{bmatrix} x_f, y_f, \phi_f, \psi_{Lf}, \psi_{Rf} \end{bmatrix} = \begin{bmatrix} 2, 5, 0, \text{free}, \text{free} \end{bmatrix}.$$

The robot accomplishes this task in 3.40 seconds and the squared sum of the control effort is $5.35 \text{ N}^2 \text{ m}^2 \text{ s}$. As can be seen in figures 3.49 and 3.51 the CM of the robot traces a straight

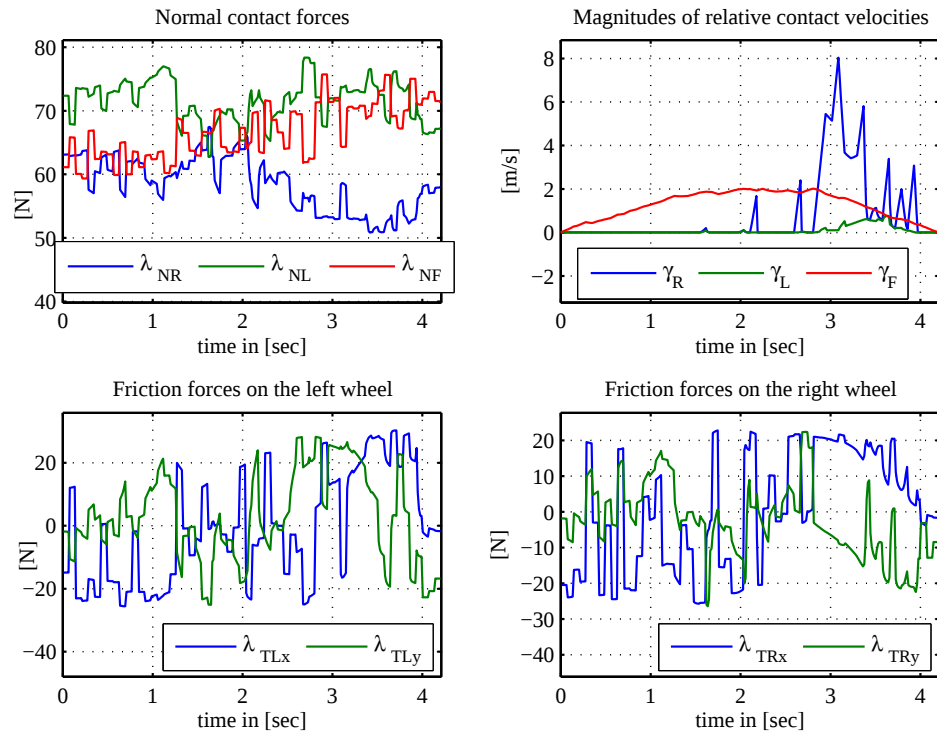


Figure 3.42: Case B: Contact forces and contact relative velocities.

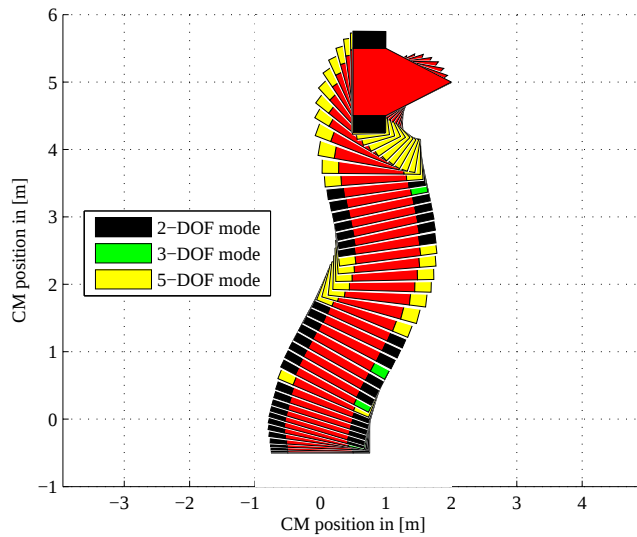


Figure 3.43: Case A: Number of DOF during the maneuver.

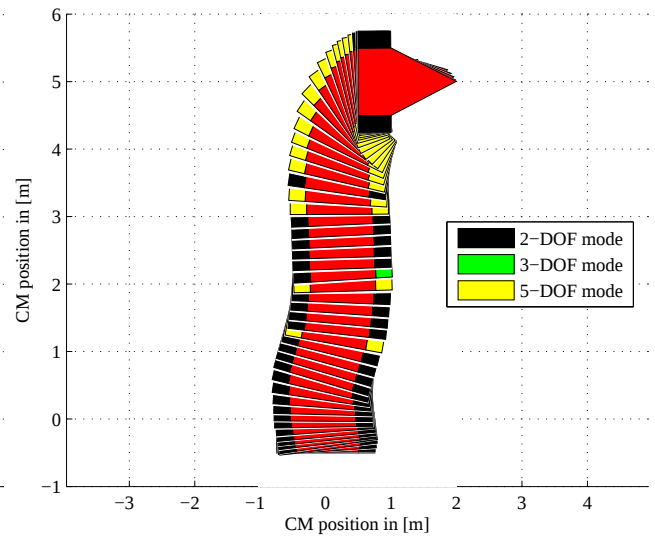


Figure 3.44: Case B: Number of DOF during the maneuver.

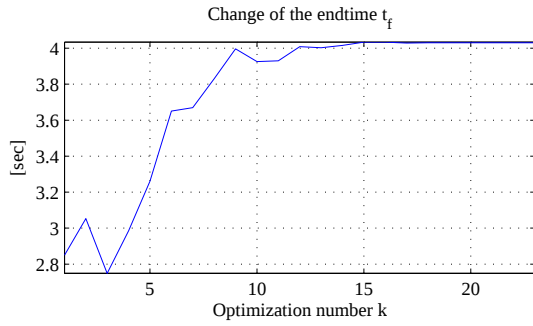


Figure 3.45: Case A: The final time t_f during successive minimisations.

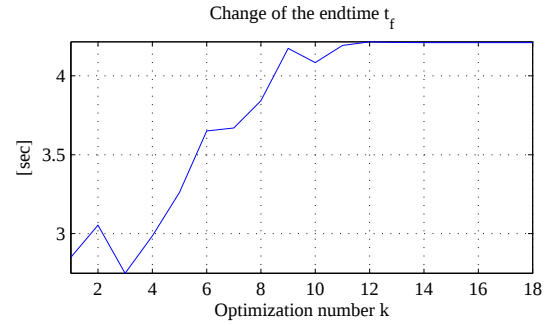


Figure 3.46: Case B: The final time t_f during successive minimisations.

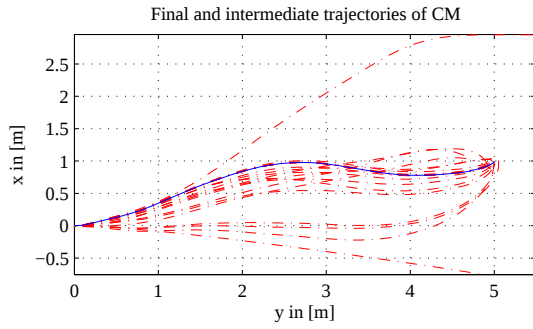


Figure 3.47: Case A: The evolution of the trajectory of the CM.

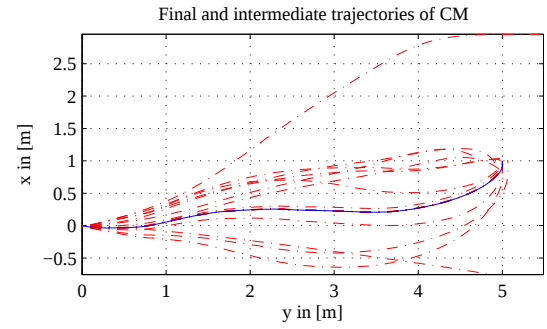


Figure 3.48: Case B: The evolution of the trajectory of the CM.

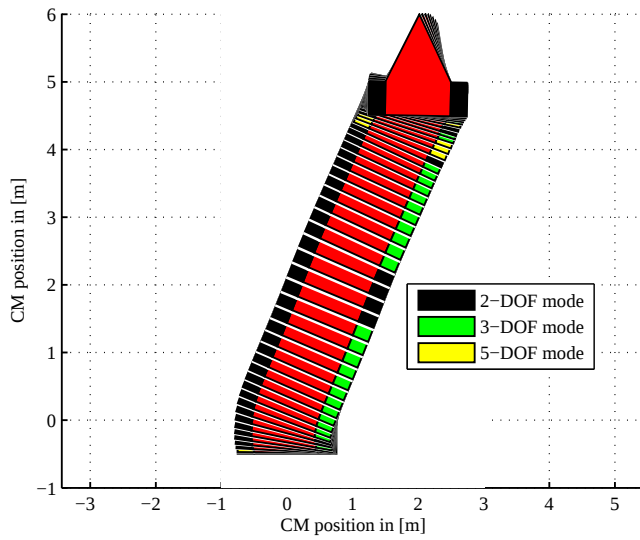


Figure 3.49: Case C: Number of DOF during the maneuver.

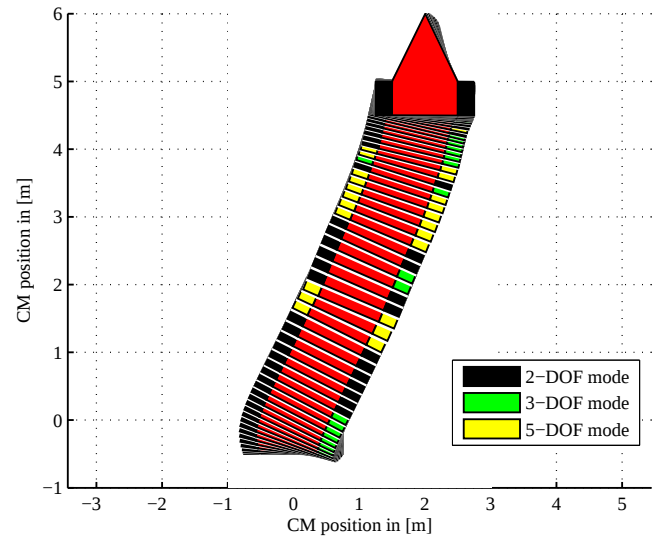


Figure 3.50: Case D: Number of DOF during the maneuver.

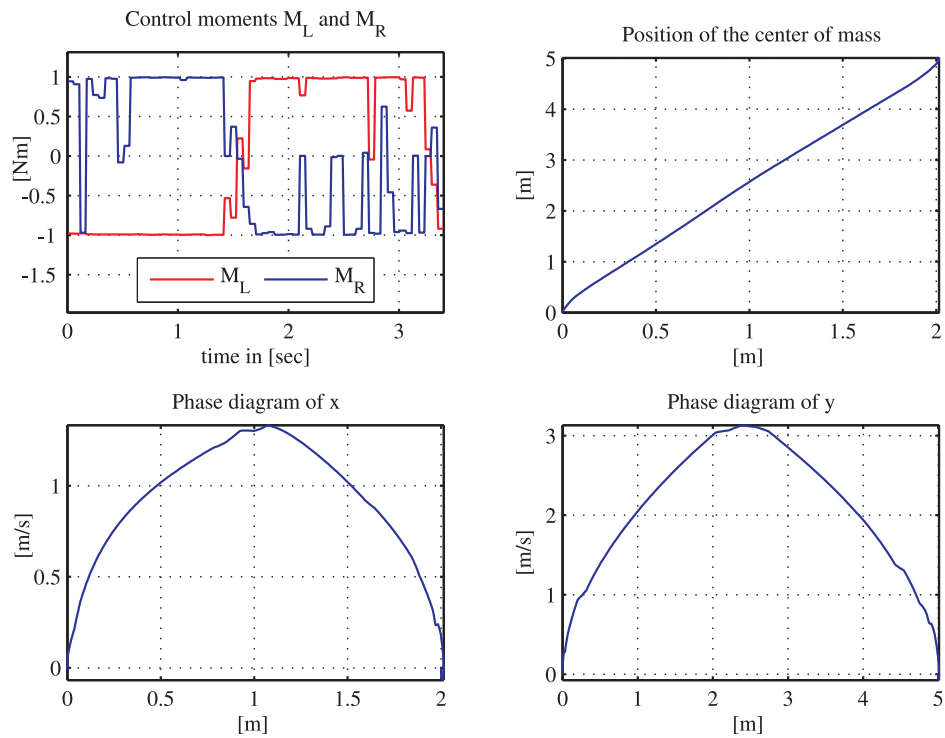


Figure 3.51: Case C: Control Moments and phases of chassis DOF.

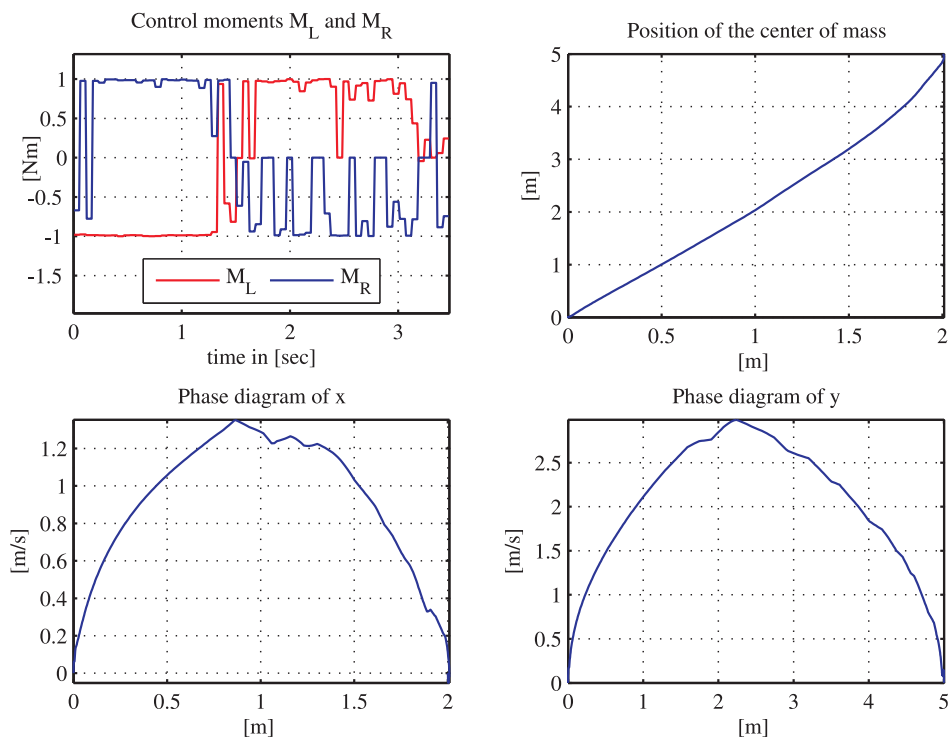


Figure 3.52: Case D: Control Moments and phases of chassis DOF.

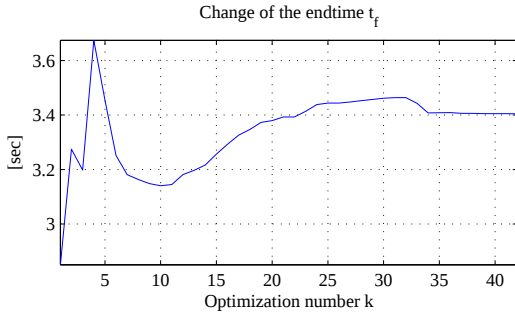
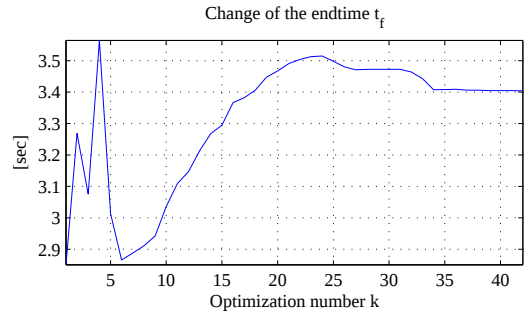
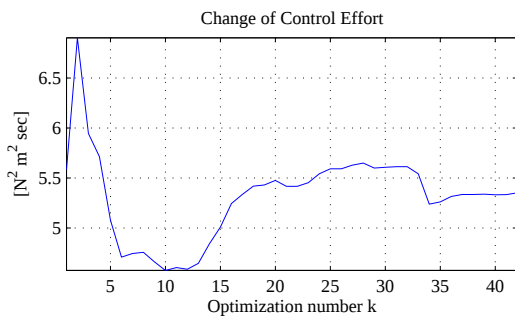
Figure 3.53: Case C: Final time t_f .Figure 3.54: Case D: Final time t_f .

Figure 3.55: Case C: The control effort.

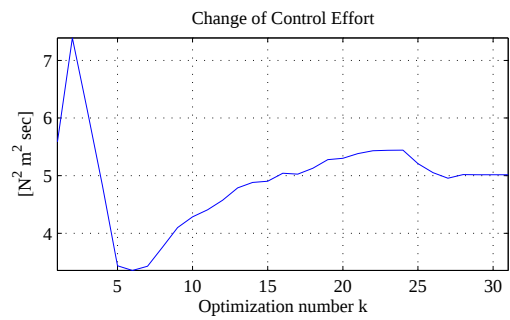


Figure 3.56: Case D: The control effort.

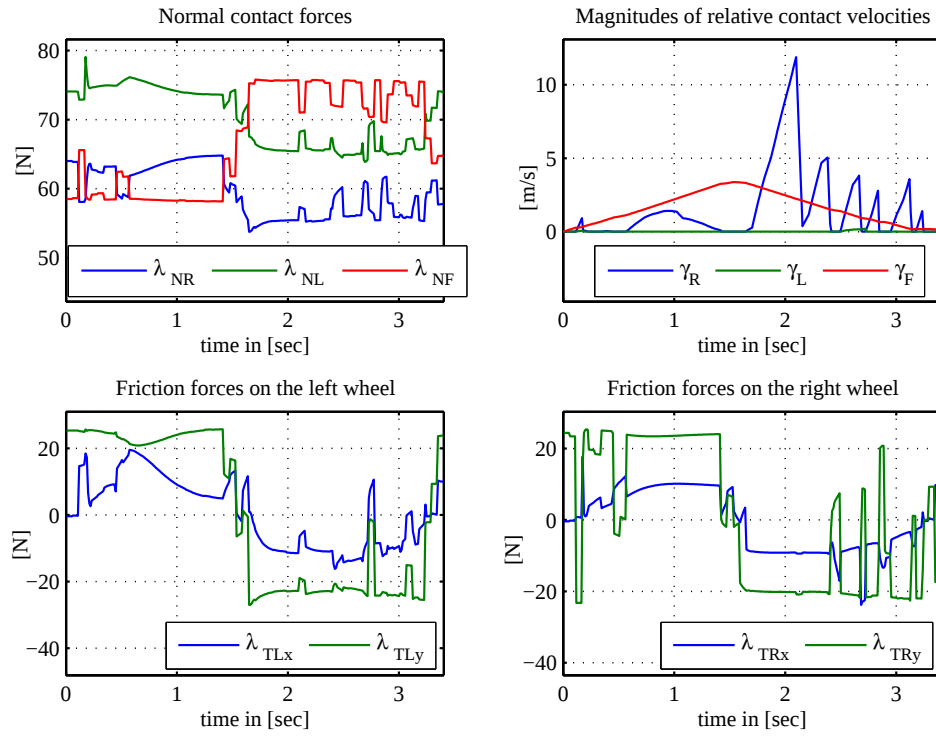


Figure 3.57: Case C: Contact forces and contact relative velocities.

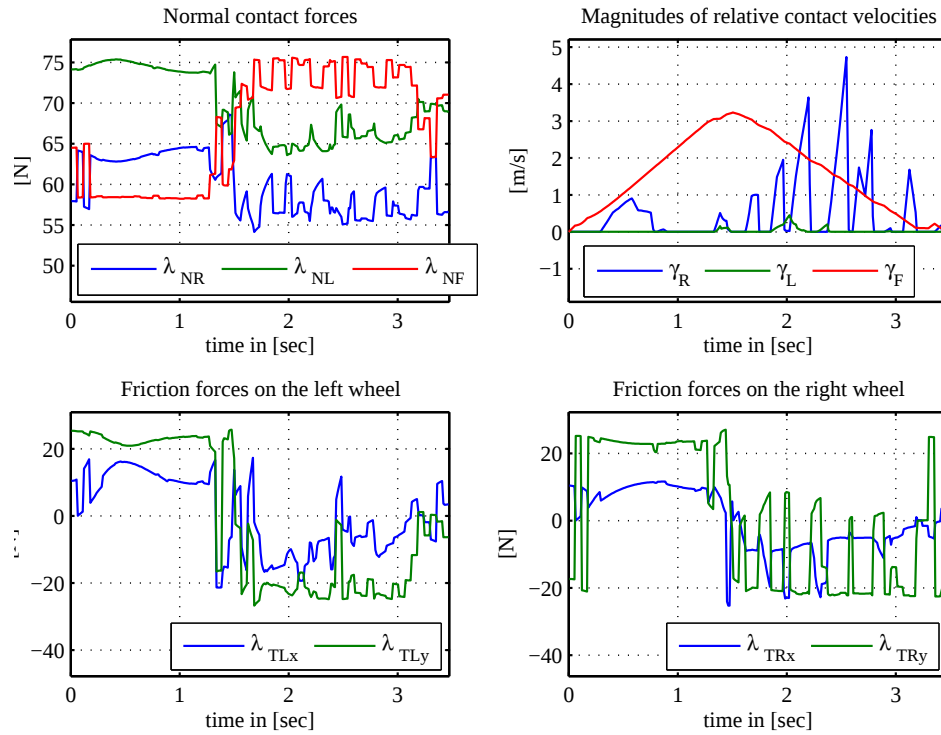


Figure 3.58: Case D: Contact forces and contact relative velocities.

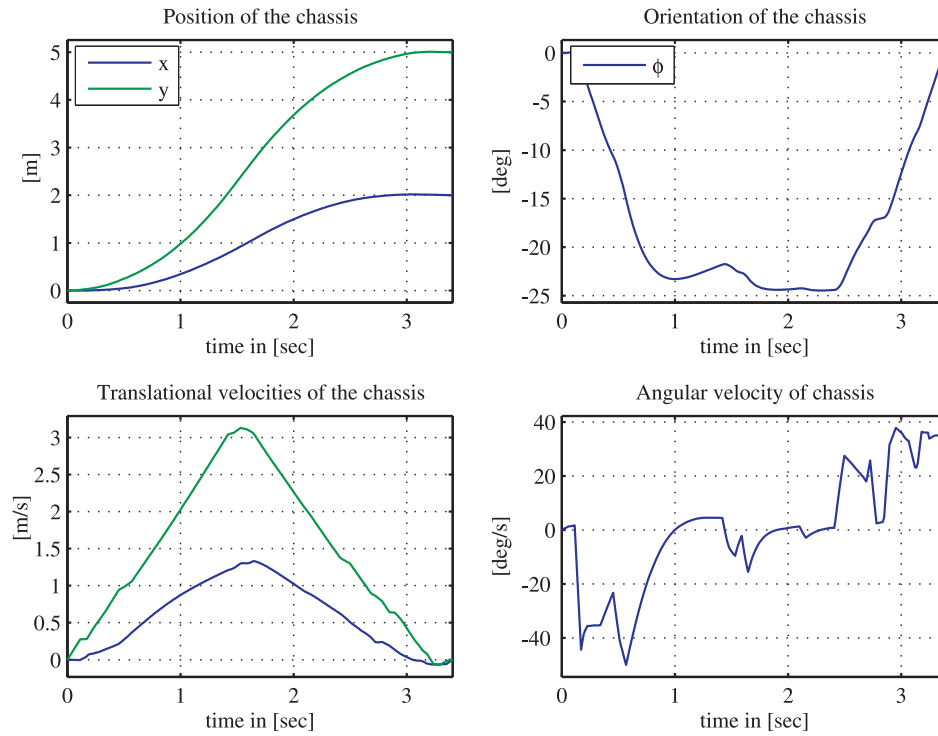


Figure 3.59: Case C: Contact forces and contact relative velocities.

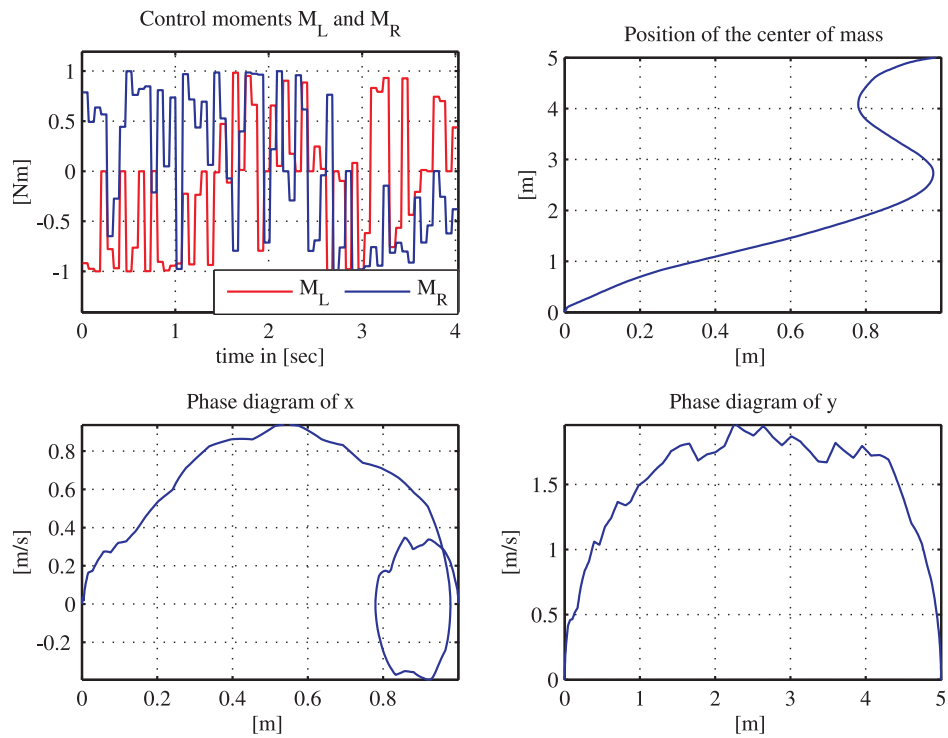


Figure 3.60: Case A: Control Moments and phases of chassis DOF.

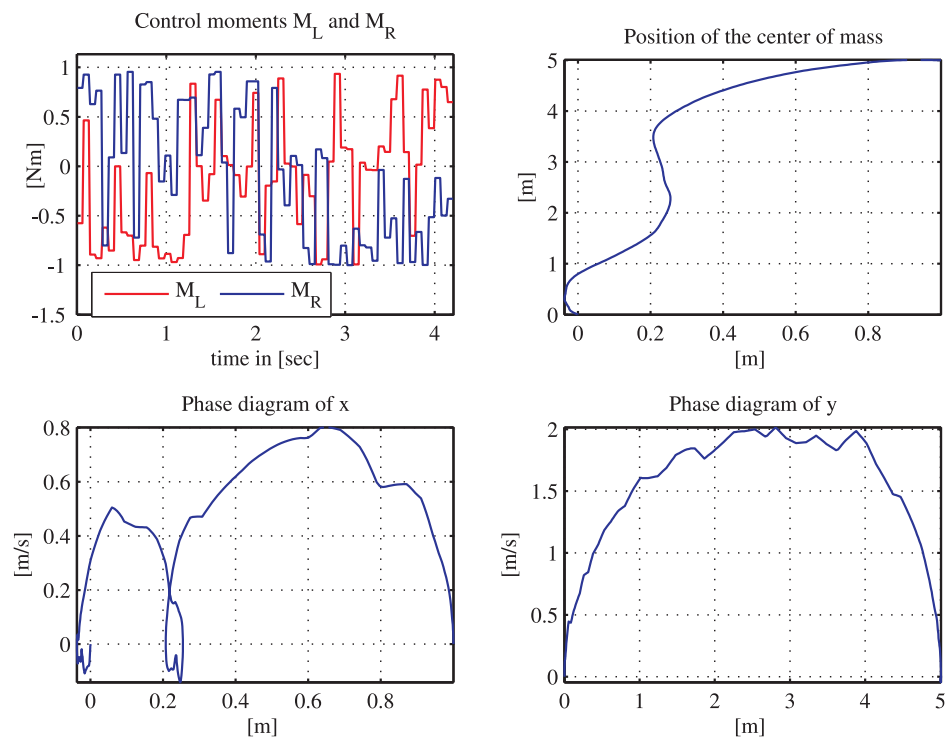


Figure 3.61: Case B: Control Moments and phases of chassis DOF.

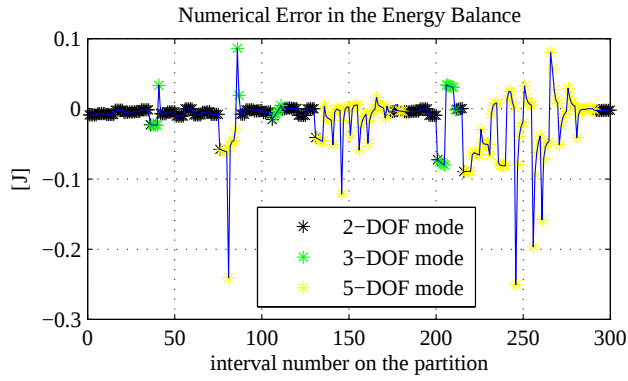
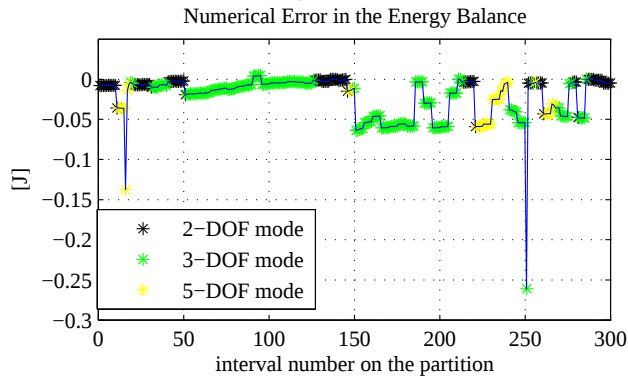
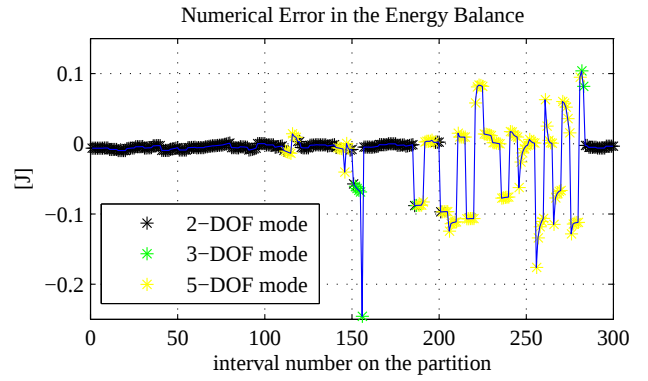
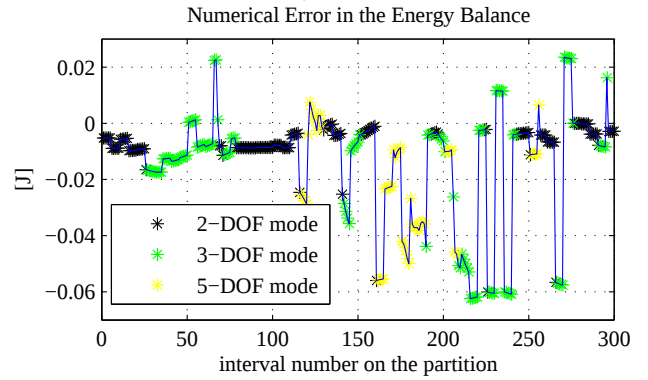
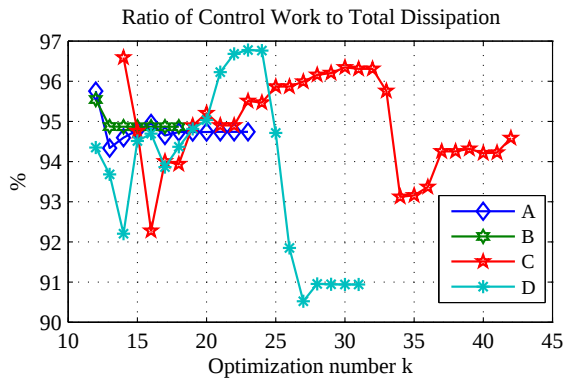
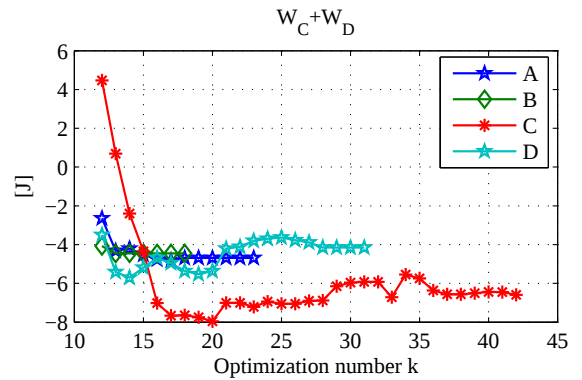
line as would be expected from a time-optimal solution. Since a straight line is the shortest distance that connects two points, a straight line trajectory between end point and final point is obtained as a result of the optimisation. One can recognise three phases of the maneuver. In the first phase of the motion the chassis is oriented approximately to the angle 21.8° which is the angle between initial and final point. In the second phase the robot accelerates on the line as can be seen in figure 3.51. In the third phase of the motion the robot reorients itself to the orientation $\phi = 0$. In the second phase of the motion due to asymmetrical loading conditions on the left and right wheels as can be inferred from the normal force trends in figure 3.57, the orientation tends to deviate from the set course. Since the chassis degrees of freedom are steered through the set-valued coupling to the control forces, the system undergoes successive stick-slip transitions on the right wheel in order to maintain orientation, which is depicted in figure 3.57 in the trends of $|\gamma_R|$, $|\gamma_L|$ and $|\gamma_F|$. It is obvious that given enough time that the system would find a solution that always moves in the 2-DOF mode, but as the final time is reduced and the control forces are driven to the allowable limits it is not possible for the system to stay always in sticking with both wheels.

3.8.4 Case D

In maneuver D the task is to reach the following position $\mathbf{q}_d$ control-effort optimally:

$$\begin{bmatrix} x_f, y_f, \phi_f, \psi_{Lf}, \psi_{Rf} \end{bmatrix} = \begin{bmatrix} 2, 5, 0, \text{free}, \text{free} \end{bmatrix}.$$

As can be seen in figures 3.50 and 3.52 the CM of the robot traces a relatively straight line and the dissipative modes are avoided during acceleration and occur only during deceleration. The robot accomplishes this task in 3.47 seconds and the squared sum of the control effort is $5.02 \text{ N}^2 \text{ m}^2 \text{ s}$. A similar three phase structure for the maneuver D is recognisable as in maneuver C but the robot spends less effort to trace a straight line, which would require intensive course correction actions. In order to save up control effort, the system makes use of frictional dissipation in order to brake down the system velocity and moves longer in the 5-DOF mode in the deceleration of the system. In comparison to the time-optimal maneuver C, it moves during the braking phase in dissipative modes which can also be seen in the trends of the magnitudes of the relative contact velocities of figures 3.57 and 3.58. The evolution of final time t_f over the successive minimisation of the augmented Lagrangian in both cases are depicted in figures 3.53 and 3.54, respectively. The evolution of total control effort over the successive minimisation of the augmented Lagrangian in both cases are depicted in figures 3.55 and 3.56, respectively.

Figure 3.62: ΔT_{err}^j for Maneuver A.Figure 3.64: ΔT_{err}^j for Maneuver C.Figure 3.63: ΔT_{err}^j for Maneuver B.Figure 3.65: ΔT_{err}^j for Maneuver D.Figure 3.66: $\frac{W_C}{W_D}$ for Maneuvers in intermediate Optimisation stages.Figure 3.67: $W_C + W_D$ for Maneuvers in intermediate Optimisation stages.

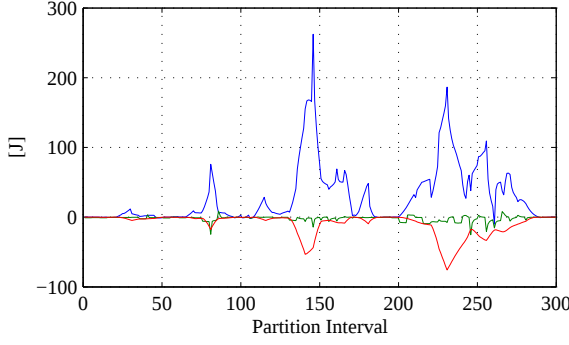


Figure 3.68: Case A: Errors in Energy.

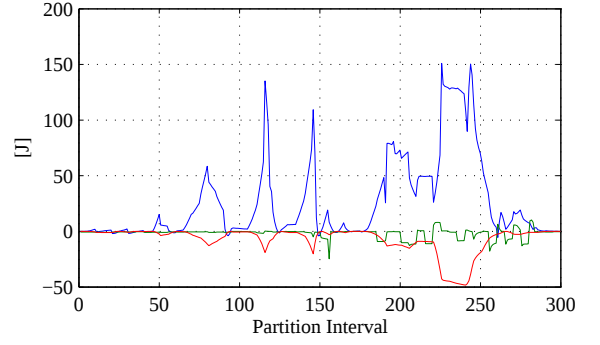


Figure 3.69: Case B: Errors in Energy.

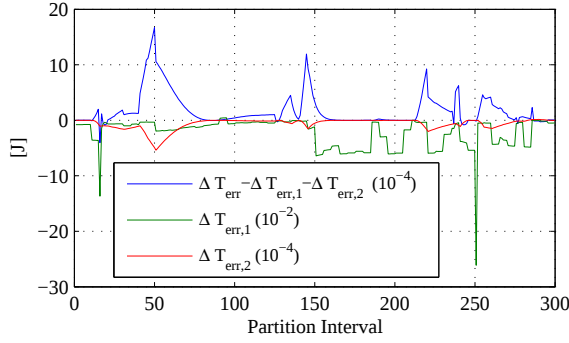


Figure 3.70: Case C: Errors in Energy.

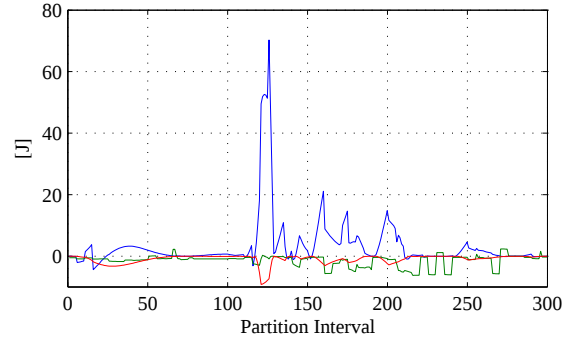


Figure 3.71: Case D: Errors in Energy.

3.8.5 Simplectic and Dissipative Properties of the Sweeping Process

The advantages of Moreau's discretisation scheme have a price because it introduces a certain error in the energy balance. The Lagrangian of the switching Lagrangian system is defined as:

$$L(\mathbf{q}, \mathbf{u}) = T(\mathbf{q}, \mathbf{u}) - V(\mathbf{q}). \quad (3.274)$$

Here $T(\mathbf{q}, \mathbf{u})$ denotes the total kinetic, and $V(\mathbf{q})$ the total smooth potential energy of the system. The total energy of the system is:

$$H(\mathbf{q}, \mathbf{u}) = T(\mathbf{q}, \mathbf{u}) + V(\mathbf{q}). \quad (3.275)$$

The differential measure of the energy H is given by:

$$dH(\mathbf{q}, \mathbf{u}) = \frac{dH}{dt} dt + (T^+ - T^-) d\sigma. \quad (3.276)$$

If the Lebesgue-Stieltjes Integral of the differential measure of the kinetic energy over an atomic time instant is evaluated then one obtains:

$$\int_{\{t_i\}} dT = T^+ - T^-. \quad (3.277)$$

The Borel measurable part of H is related to the jump in kinetic energy:

$$T^+ - T^- = \frac{1}{2} \langle \mathbf{u}^+, \mathbf{M}(\mathbf{q}) \mathbf{u}^+ \rangle - \frac{1}{2} \langle \mathbf{u}^-, \mathbf{M}(\mathbf{q}) \mathbf{u}^- \rangle, \quad (3.278)$$

and is nonzero if and only if there is an impulsive action that induces a jump in the generalised velocities. For the class of switching Lagrangian systems defined by relations (3.224) to (3.226) the differential measure of total energy is equal to the sum of the differential measure of the work of control forces dW_C and the differential measure of the dissipative work dW_D as stated below:

$$dH = dW_C + dW_D. \quad (3.279)$$

Here dW_C and dW_D are defined by:

$$dW_C = \langle \mathbf{u}^-, \mathbf{B}(\mathbf{q}) \boldsymbol{\tau}^- \rangle dt \quad (3.280)$$

and

$$dW_D = \sum_{\forall i \in \mathcal{I}_S} \left\langle \frac{1}{2} (\gamma_{T_i}^- + \gamma_{T_i}^+), d\Lambda_{T_i} \right\rangle, \quad (3.281)$$

respectively. The difference between the kinetic energies at successive discretisation points is given by:

$$\Delta T^j = T^{j+1} - T^j = \langle \mathbf{u}^{j+1}, \mathbf{M}(\mathbf{q}^{j+1}) \mathbf{u}^{j+1} \rangle - \langle \mathbf{u}^j, \mathbf{M}(\mathbf{q}^j) \mathbf{u}^j \rangle \quad (3.282)$$

$$\approx \frac{1}{2} \langle \mathbf{u}^{j+1}, \mathbf{M}(\mathbf{q}_m^j) \mathbf{u}^{j+1} \rangle - \frac{1}{2} \langle \mathbf{u}^j, \mathbf{M}(\mathbf{q}_m^j) \mathbf{u}^j \rangle \quad (3.283)$$

$$= \frac{1}{2} \langle \mathbf{u}^{j+1} - \mathbf{u}^j, \mathbf{M}(\mathbf{q}_m^j) (\mathbf{u}^{j+1} + \mathbf{u}^j) \rangle \quad (3.284)$$

and the difference in potential energy is approximated by:

$$\Delta V^j = V^{j+1} - V^j \approx \langle \mathbf{u}^{j+1}, \mathbf{h}_{\text{pot}}(\mathbf{q}^{j+1}) \rangle \frac{\Delta t}{2} + \langle \mathbf{u}^j, \mathbf{h}_{\text{pot}}(\mathbf{q}^j) \rangle \frac{\Delta t}{2} \quad (3.285)$$

$$= \langle \mathbf{u}^j + \mathbf{u}^{j+1}, \mathbf{h}_{\text{pot}}(\mathbf{q}_m) \rangle \frac{\Delta t}{2} = \langle \mathbf{q}^{j+1} - \mathbf{q}^j, \mathbf{h}_{\text{pot}}(\mathbf{q}_m) \rangle. \quad (3.286)$$

Here the vector $\mathbf{h}(\mathbf{q}, \mathbf{u})$ is separated in its potential and nonpotential parts:

$$\mathbf{h}(\mathbf{q}, \mathbf{u}) = \mathbf{h}_{\text{nonpot}}(\mathbf{q}, \mathbf{u}) + \mathbf{h}_{\text{pot}}(\mathbf{q}). \quad (3.287)$$

The discretized approximations of the differential measures of control work and dissipative work are given by:

$$\Delta W_C^j = \langle \mathbf{u}^j, \mathbf{B}(\mathbf{q}_m^j) \boldsymbol{\tau}^j \rangle \Delta t, \quad (3.288)$$

$$\Delta W_D^j = \sum_{\forall i \in \mathcal{I}_S} \left\langle \frac{1}{2} (\gamma_{T_i}^j + \gamma_{T_i}^{j+1}), \Lambda_{T_i}^j \right\rangle. \quad (3.289)$$

The evaluation of the dissipation unifies in it the impulse and the force character. Though $\mathbf{\Lambda}_{T_i}$ is an impulse and is seen mathematically as a distribution on its support, by assuming that it has a constant density on its support, which in this case has the measure Δt , is equivalently represented as $\mathbf{\Lambda}_{T_i} = \boldsymbol{\lambda}_{T_i} \Delta t$. Here $\boldsymbol{\lambda}_{T_i}$ is a force and the value of the constant density on the support. By making use of this distributional approach, the work of the frictional contact impulses can be evaluated like forces. The question that poses itself is now naturally to define the instant at which this force acts in order to obtain its work contribution by dual pairing with a suitable velocity. Here one notices that this force equally distributed over the support and the midpoint velocity on the support as used in ΔW_D^j turns out to be suitable. The energy balance over an interval then reads:

$$\Delta V^j + \Delta T^j - \Delta W_C^j - \Delta W_D^j = \Delta H_{\text{err}}^j. \quad (3.290)$$

Insertion of the discretized form of the measure-differential inclusion as given below in the expression for ΔT^j :

$$\mathbf{u}^{j+1} - \mathbf{u}^j = \mathbf{M}^{-1}(\mathbf{q}_m^j) \left(\mathbf{h}(\mathbf{q}_m^j, \mathbf{u}^j) \Delta t + \sum_{\forall i \in \mathcal{I}_S} \mathbf{W}_{T_i}(\mathbf{q}^m) \mathbf{\Lambda}_{T_i} + \mathbf{B}(\mathbf{q}^m) \boldsymbol{\tau}^j \Delta t \right), \quad (3.291)$$

and considering $\boldsymbol{\gamma}_{T_i}^j + \boldsymbol{\gamma}_{T_i}^{j+1} = \mathbf{W}_{T_i}^T(\mathbf{q}^m) (\mathbf{u}_{T_i}^j + \mathbf{u}_{T_i}^{j+1})$ reveals the expression for the error ΔH_{err}^j for interval j as:

$$\begin{aligned} \Delta H_{\text{err}}^j &\approx \frac{\Delta t}{2} (\langle \mathbf{u}^{j+1} - \mathbf{u}^j, \mathbf{B}(\mathbf{q}_m^j) \boldsymbol{\tau}^j \rangle + \langle \mathbf{u}^{j+1} + \mathbf{u}^j, \mathbf{h}_{\text{nonpot}}(\mathbf{q}_m^j, \mathbf{u}^j) \rangle) \\ &= \Delta H_{\text{err},1}^j + \Delta H_{\text{err},2}^j. \end{aligned} \quad (3.292)$$

The expressions $\Delta H_{\text{err},1}^j$ and $\Delta H_{\text{err},2}^j$ are defined as:

$$\Delta H_{\text{err},1}^j = \frac{\Delta t}{2} \langle \mathbf{u}^{j+1} - \mathbf{u}^j, \mathbf{B}(\mathbf{q}_m^j) \boldsymbol{\tau}^j \rangle \quad (3.293)$$

and

$$\Delta H_{\text{err},2}^j = \frac{\Delta t}{2} \langle \mathbf{u}^{j+1} + \mathbf{u}^j, \mathbf{h}_{\text{nonpot}}(\mathbf{q}_m^j, \mathbf{u}^j) \rangle, \quad (3.294)$$

respectively. The analysis of the energy balance for the differential-drive robot provides a visualisation of the inherent energy deficiency. For a mechanical system like the differential-drive robot where there are no changes in the gravitational potential and $V(\mathbf{q})$ is constant throughout, one has:

$$\int_{t_0}^{t_f} dT = \int_{t_0}^{t_f} dW_C + dW_D = 0, \quad (3.295)$$

if the generalised velocities are zero at t_0 and t_f . In this case the error in the energy balance ΔH_{err}^j is set equal to ΔT_{err}^j .

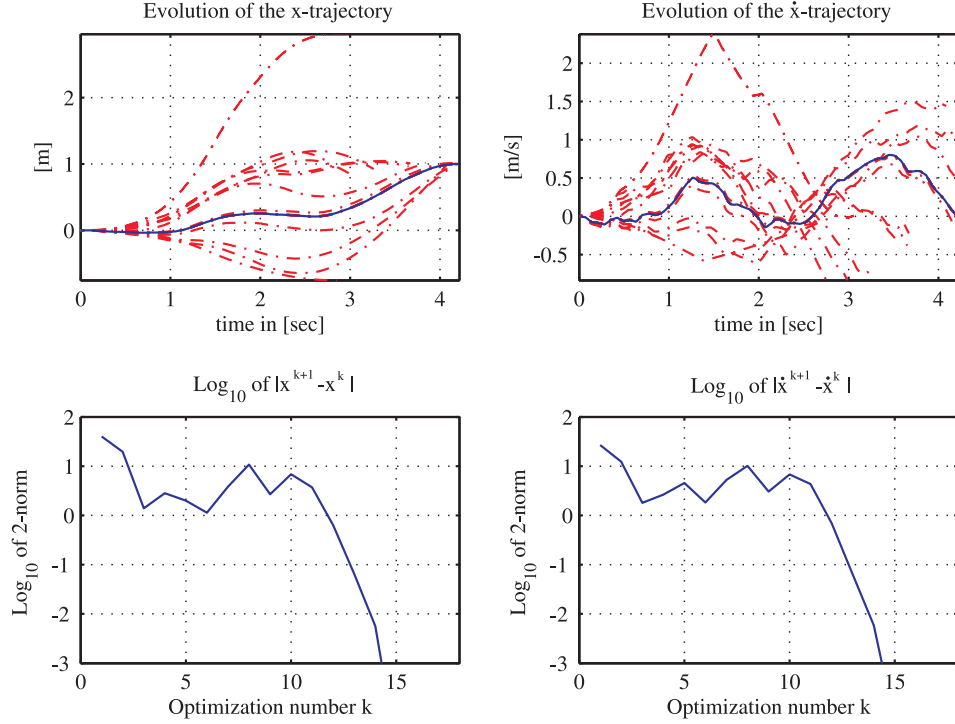


Figure 3.72: Case B: Evolution and Convergence Behaviour of x and $\dot{x}$ trajectories.

Figure 3.66 indicates the ratio $\frac{W_C}{W_D}$ for the maneuvers A, B, C, D in the intermediate changes of the optimisation, whereas figure 3.67 depicts the energy deficiency in Joules for the same cases. Figures 3.62 to 3.65 show for the cases A, B, C and D the numerical values of ΔH_{err}^j and the color code indicates in which mode the robot moves for given energy deficiency. Figures 3.68 to 3.71 reflect the magnitude plots of $\Delta T_{err,1}^j$ and $\Delta T_{err,2}^j$ as calculated by equations (3.293) and (3.294), and $\Delta T_{err}^j - \Delta T_{err,1}^j - \Delta T_{err,2}^j$ where ΔT_{err}^j is obtained by (3.290) by setting $\Delta T_{err}^j = \Delta H_{err}^j$.

3.8.6 Convergence Behaviour of Case B

In this subsection the evolution of various trajectories for Case B is presented. The optimal result in Case B is obtained after 15 successive minimisation stages. Quantitatively the evolution of the phase trajectories in Maneuver B as visualized in figure 3.77 is an indication of the global optimizing behaviour of the proposed algorithm. Figures 3.72, 3.73, 3.74, 3.75 and 3.76 show the evolution of respective trajectories and the logarithm of the distance of successive trajectories together. It is characteristic that at the solution as a result of the convergent behaviour of the algorithm the distance between successive trajectories gets very small. This convergent behaviour is seen in the evolution of the generalised positions and velocities in figures 3.72, 3.73, 3.74, 3.75 and 3.76.

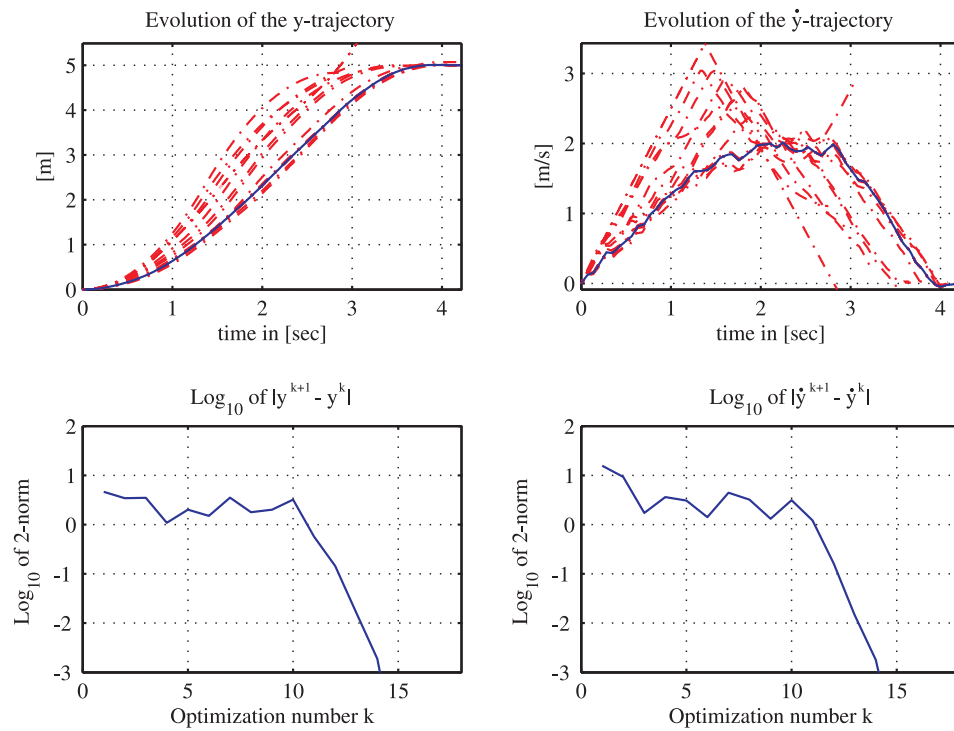


Figure 3.73: Case B: Evolution and Convergence Behaviour of y and $\dot{y}$ trajectories.

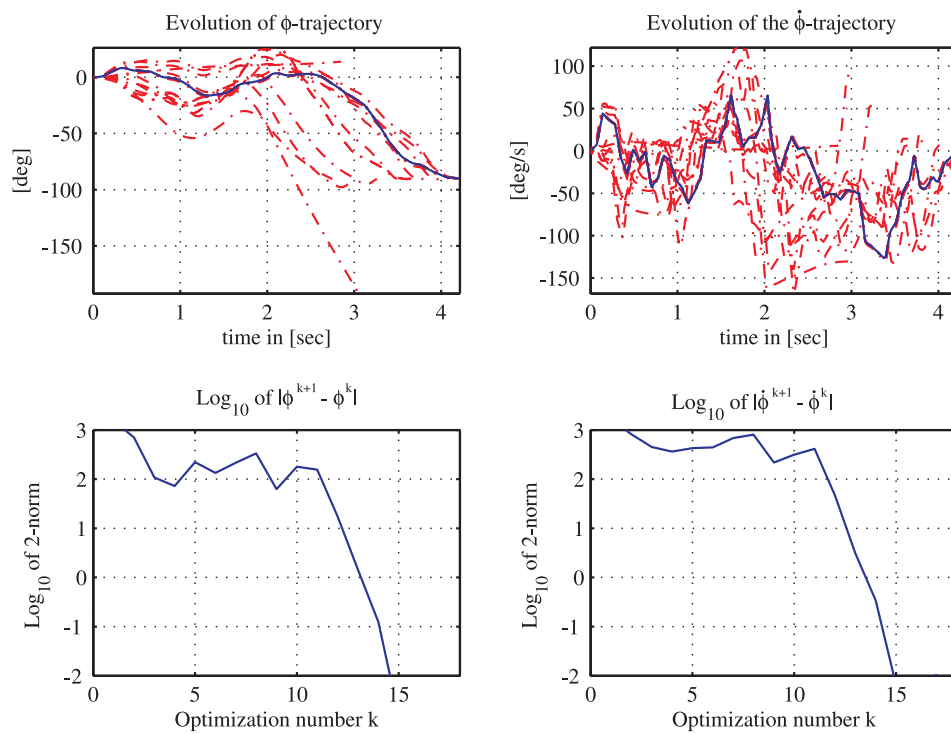
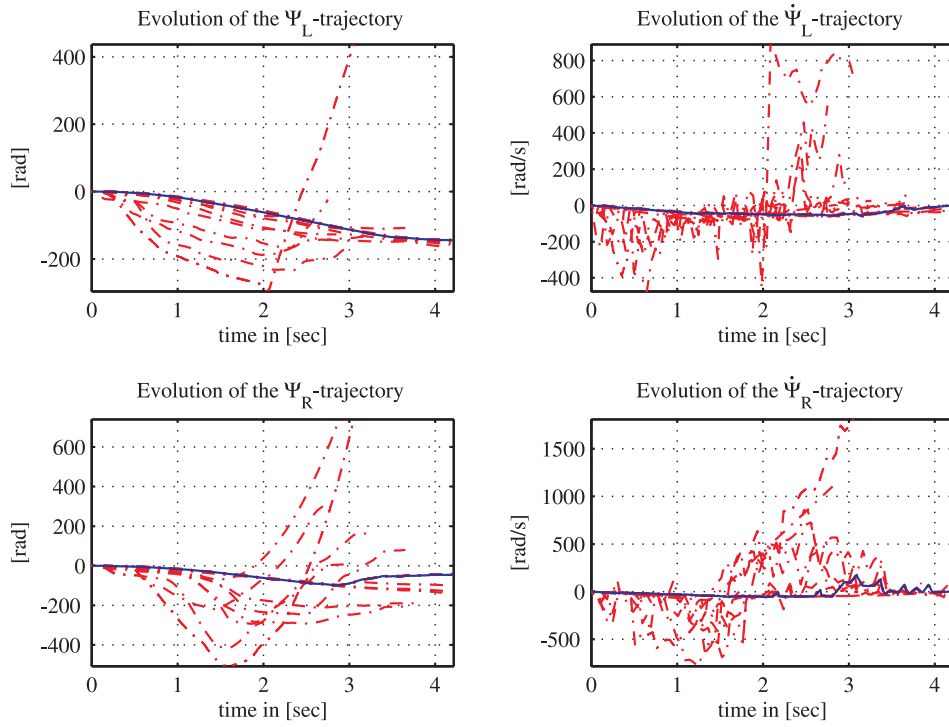
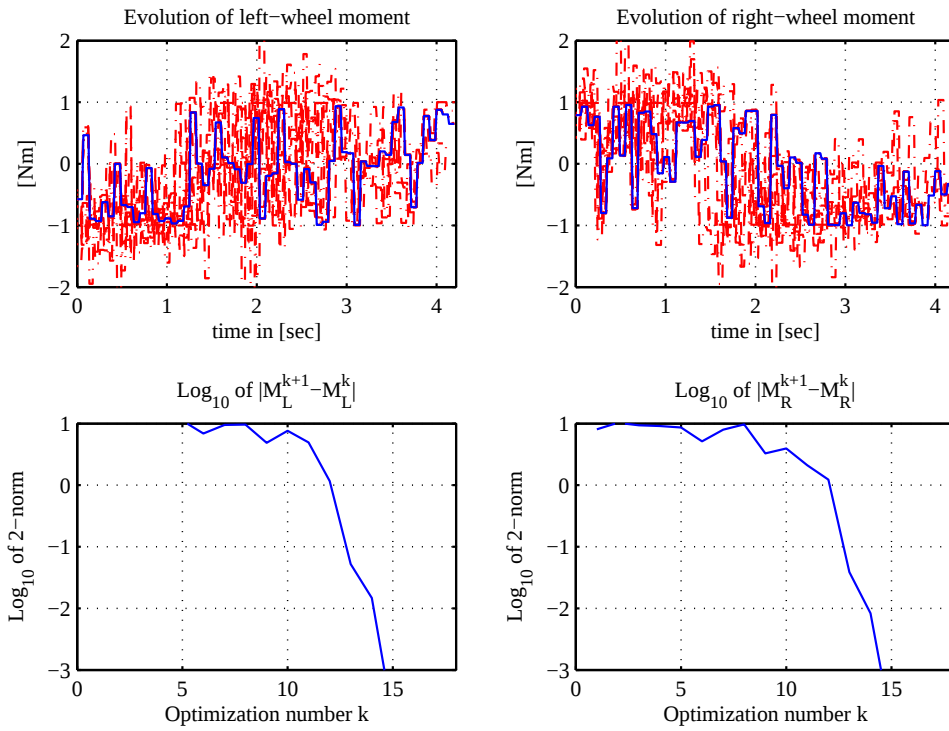


Figure 3.74: Case B: Evolution and Convergence Behaviour of ϕ and $\dot{\phi}$ trajectories.

Figure 3.75: Case B: Evolution of Ψ_R , Ψ_L , $\dot{\Psi}_R$ and $\dot{\Psi}_L$ trajectories.Figure 3.76: Case B: Evolution and Convergence Behaviour of M_L and M_R trajectories.

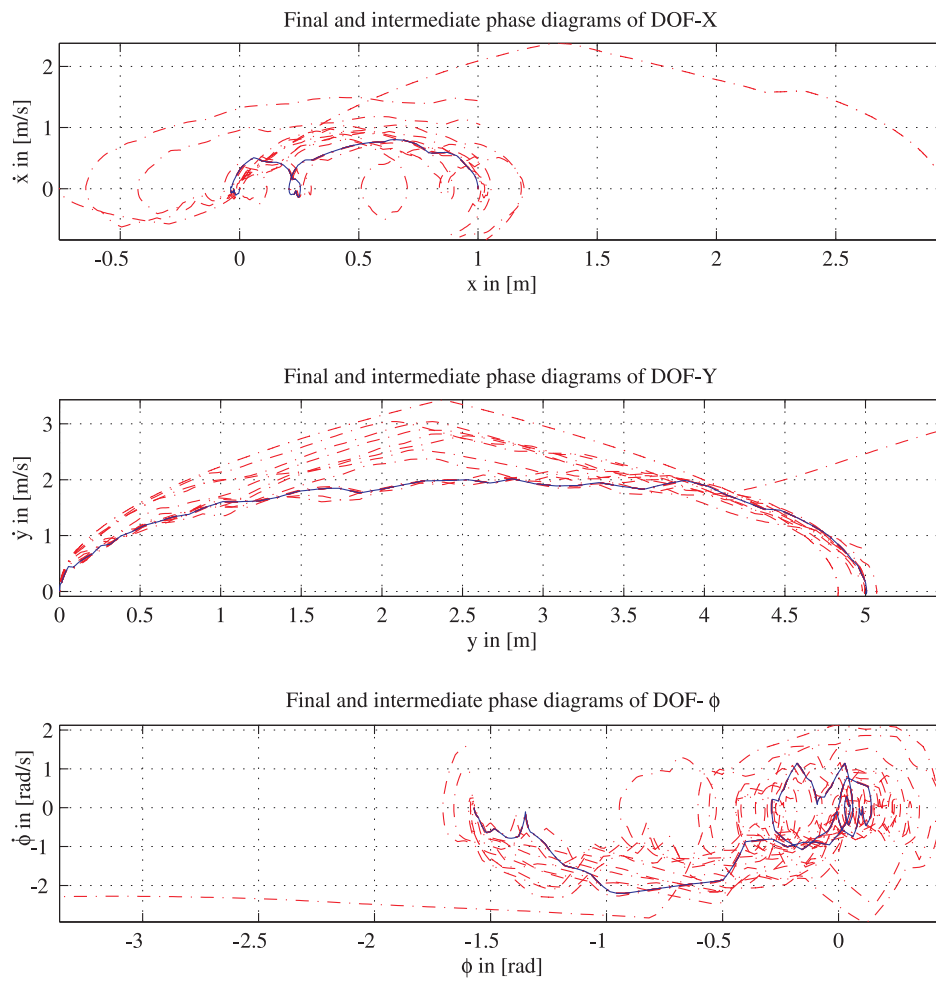


Figure 3.77: Case B: Evolutions of DOF x, y and ϕ as phase diagrams.

Chapter 4

Internal Boundary Variations, Discontinuous Transversality Conditions and Optimal Control

In this work, necessary conditions for the impulsive optimal control of multibody mechanical systems are stated. The conditions are obtained by the application subdifferential calculus techniques to extended-valued lower semi-continuous generalised Bolza functional that is evaluated on multiple intervals. Contrary to the approach in literature so far, an instant at which possibly impulsive transition takes place is considered as a Lebesgue negligible instant. This approach is in comparison to other impulsive necessary conditions consistent with mainstream hybrid system modeling methods in which transitions happen instantaneously. The necessary conditions provide necessary criteria for the determination of optimal transition times and locations. The consideration of certain type of variations at the boundaries give birth to the concepts of internal boundary variations and discontinuous transversality conditions. The concepts are developed by the author and are presented and discussed in [110] and [112] with applications to optimal control. In this work, a characterisation of these concepts in terms of upper and lower subderivatives to the extended-valued lower-semicontinuous value functional under several regularity assumptions is given. The properties of the transition sets are discussed.

4.1 Introduction

An impact in mechanics is defined as a discontinuity in the generalised velocities of a mechanical system which is induced by impulsive forces, therefore optimal control of such impulsive systems inevitably encompasses optimal control with discontinuous generalised velocities. The main issue in the optimal control of impactive mechanical systems has been the blending of impact mechanics with impulsive optimal control. The crux in the derivation of these necessary

conditions is to handle joint discontinuity of the state and the dual state on a Lebesgue negligible interval. In the framework of integration theory, this has long been recognised as a problem if state and costate should become concurrently discontinuous as has been addressed in [64] by Moreau and [82] by Rockafellar. Rockafellar studied in [82] the discontinuity of the dual state in constrained convex optimal control problems but dispensed of attacking the problem concurrent discontinuity of state and costate. Moreau gave in [64] partial integration formulas for differential measures in general bilinear forms. In [68] Murray studies the extension and existence theorems of problems in calculus of variation to the setting when impulsive controls are applied and state discontinuity occurs. He bases his work on [82], and outlines in his motivation that jumps in the states may occur due to constraints on the dual dynamics which are reached by the costate and considers applications in the field of economics. In [18] several classes of impulsive Lagrangian systems are studied by Bressan. The main focus is impulses generated by sudden parameter changes such as inertial parameters that affect the momentum balance, or impulses arising due to structure of constraints of a mechanical system. A certain class of impulsive systems that resemble discontinuous diffusion processes are treated in [12] by Bensoussan.

In what follows next, the new concepts required to deal with this specific problem are introduced. In order to overcome the difficulties arising from joint discontinuity of state and costate, the instant of impulsive control action where discontinuity in the generalised velocities occur is considered as an internal boundary in the time domain. In [112] by Yunt, the concept of internal boundary variations are introduced literally, and as an application a theorem that states the necessary conditions for the impulsive time-optimal control of finite-dimensional Lagrangian systems is stated. In the framework of these concepts, philosophically, the instant of state discontinuity constitutes an internal boundary in the optimal control problem. The essential idea is thus to consider every point of the domain where continuity and differentiability ceases to exist, as a boundary of the problem. By introducing a boundary at an instant of a discontinuity, one has to notice that it has bilateral character, in the sense that the boundary constitutes an upper boundary for one segment of the interval whereas for the other segment a lower boundary in the time domain. The necessity that at a location of transition several conditions have to be fulfilled, gives rise to the idea of some sort of transversality conditions if one begins to consider an instant of discontinuity as a two-sided boundary where two arcs are "connected" discontinuously. This dependence is embedded in the concept of internal boundary variations. In order to obtain criteria for the optimality of the transition position, transition pre-, and post-transition generalised velocities, transition time and impulsive control, variations in these entities need to be considered, which represent in the setting of this work the internal boundary variations. At the boundaries of the time domain, the pre-transition state variations are considered separately from the post-transition variations. The absolute continuity of the generalised

positions means that the total variation of the generalised positions at the pre-transition and post-transition instants are equal. The pre-transition and post-transition variations are inter-related by the transition conditions which can be seen as the bases of transversality conditions that join two trajectories discontinuously. The transition conditions are introduced symmetrically with respect to pre-, and post-transition states. The transition conditions are of two types, namely, the impact equation and the constitutive impact laws. The impact equations relate the discontinuity in the impulse of the Lagrangian system to the impulsive forces/controls. The impact law (i.e. the moreau-newton impact law), however, is a constitutive law which is chosen depending on the modeling approach preferred. As a case study, in reference [111] the blocking of some DOF of an underactuated manipulator by tangential fully-inelastic impact is discussed and the necessary conditions are stated. In [110] the necessary conditions for the impulsive optimal control of Lagrangian systems in the Hamiltonian framework is investigated. By the application of subdifferential calculus techniques to extended-valued lower semi-continuous functionals, necessary conditions are obtained. In publications of R. T. Rockafellar such [84] and [85] a summary of the rules in subdifferential calculus are provided, which is one of the most flourishing branches of mathematics.

4.2 Preliminaries

Definition 4.1.1 The Proximal Normal Vector Given a closed set $\mathcal{C} \in \mathbb{R}^k$ and a point $\bar{\mathbf{x}} \in \mathcal{C}$, a vector $\mathbf{et} \in \mathbb{R}^k$ is called to be a proximal normal vector to $\mathcal{C}$ at $\bar{\mathbf{x}}$ if there exists $M \geq 0$ such that

$$\langle \boldsymbol{\eta}, \mathbf{x} - \bar{\mathbf{x}} \rangle \leq M |\mathbf{x} - \bar{\mathbf{x}}|^2, \forall \mathbf{x} \in \mathcal{C}. \quad (4.1)$$

Definition 4.1.2 The Proximal Normal Cone The cone of all proximal normal vectors $\boldsymbol{\eta}$ to $\mathcal{C}$ is called the proximal normal cone to $\mathcal{C}$ at $\bar{\mathbf{x}}$ and is denoted by $\mathcal{N}_{\mathcal{C}}^P(\bar{\mathbf{x}})$:

$$\mathcal{N}_{\mathcal{C}}^P(\bar{\mathbf{x}}) := \{ \boldsymbol{\eta} \mid \langle \boldsymbol{\eta}, \mathbf{x} - \bar{\mathbf{x}} \rangle \leq M |\mathbf{x} - \bar{\mathbf{x}}|^2, \forall \mathbf{x} \in \mathcal{C}, \mathbf{et} \in \mathbb{R}^k : \exists M \geq 0 \}. \quad (4.2)$$

Definition 4.1.3 The Limiting Normal Vector Given a closed set $\mathcal{C} \in \mathbb{R}^k$ and a point $\bar{\mathbf{x}} \in \mathcal{C}$, a vector $\mathbf{et} \in \mathbb{R}^k$ is called to be a limiting normal vector to $\mathcal{C}$ at $\bar{\mathbf{x}} \in \mathcal{C}$ if there exist sequences $\mathbf{x}_i \xrightarrow{\mathcal{C}} \bar{\mathbf{x}}$ and $\boldsymbol{\eta}_i \rightarrow \boldsymbol{\eta}$ such that

$$\boldsymbol{\eta}_i \in \mathcal{N}_{\mathcal{C}}^P(\bar{\mathbf{x}}_i), \quad \forall i. \quad (4.3)$$

Definition 4.1.4 The Limiting Normal Cone The cone of limiting normal vectors to $\mathcal{C}$ at $\bar{\mathbf{x}}$ is denoted $\mathcal{N}_{\mathcal{C}}(\bar{\mathbf{x}})$ and is defined by:

$$\mathcal{N}_{\mathcal{C}}(\bar{\mathbf{x}}) := \{ \mathbf{et} \in \mathbb{R}^k \mid \exists \mathbf{x}_i \xrightarrow{\mathcal{C}} \bar{\mathbf{x}} \wedge \boldsymbol{\eta}_i \rightarrow \boldsymbol{\eta}, \boldsymbol{\eta}_i \in \mathcal{N}_{\mathcal{C}}^P(\bar{\mathbf{x}}_i), \quad \forall i \}. \quad (4.4)$$

Definition 4.1.5 The Strict Normal Vector and Strict Normals Given a closed set $\mathcal{C} \subset \mathbb{R}^k$ and a point $\mathbf{x} \in \mathcal{C}$, the strict normal cone to $\mathcal{C}$ at $\mathbf{x}$, written $\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x})$, is the set

$$\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x}) := \left\{ \mathbf{p} \mid \limsup_{\mathbf{y} \xrightarrow{\mathcal{C}} \mathbf{x}} \frac{\langle \mathbf{p}, \mathbf{y} - \mathbf{x} \rangle}{|\mathbf{y} - \mathbf{x}|} \leq 0 \right\}. \quad (4.5)$$

Elements in $\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x})$ are called strict normals to $\mathcal{C}$ at $\mathbf{x}$.

Some elementary properties of the cones that have been introduced are listed without proof.

Proposition 4.1.1 Take a closed set $\mathcal{C} \subset \mathbb{R}^k$ and points $\mathbf{x} \in \mathcal{C}$ and $\mathbf{p} \in \mathbb{R}^k$. Then the following assertions are equivalent:

1. $\mathbf{p} \in \mathcal{N}_{\mathcal{C}}(\mathbf{x})$,
2. there exist sequences $\mathbf{p}_i$ and $\mathbf{x}_i$ such that $\mathbf{p}_i \in \hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x}_i)$, $\mathbf{x}_i \rightarrow \mathbf{x}$, $\mathbf{p}_i \rightarrow \mathbf{p}$, $\forall i$.

Proposition 4.1.2 Take a closed set $\mathcal{C} \subset \mathbb{R}^k$ and points $\mathbf{x} \in \mathcal{C}$. Then:

1. $\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x})$, $\mathcal{N}_{\mathcal{C}}^{\mathbf{P}}(\mathbf{x})$ and $\mathcal{N}_{\mathcal{C}}(\mathbf{x})$ are all cones in $\mathbb{R}^k$, containing $\{\mathbf{0}\}$ and

$$\mathcal{N}_{\mathcal{C}}^{\mathbf{P}}(\mathbf{x}) \subset \hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x}) \subset \mathcal{N}_{\mathcal{C}}(\mathbf{x}); \quad (4.6)$$

2. $\mathcal{N}_{\mathcal{C}}^{\mathbf{P}}(\mathbf{x})$ is convex, but not necessarily closed;
3. $\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x})$ is closed and convex;
4. the set-valued mapping $\mathbf{y} \rightarrow \mathcal{N}_{\mathcal{C}}(\mathbf{y}) : \mathcal{C} \rightarrow \mathbb{R}^k$ has a closed graph, in the sense that, for any sequences $\mathbf{y}_i \xrightarrow{\mathcal{C}} \mathbf{y}$ and $\mathbf{p}_i \rightarrow \mathbf{p}$ such that $\mathbf{p}_i \in \mathcal{N}_{\mathcal{C}}(\mathbf{y}_i)$ for all i , we have $\mathbf{p} \in \mathcal{N}_{\mathcal{C}}(\mathbf{y})$.

Proposition 4.1.3 Take a closed set $\mathcal{C} \subset \mathbb{R}^k$ and a point $\mathbf{x} \in \mathcal{C}$. Then:

1. $\mathbf{x} \in \text{int}\{\mathcal{C}\}$ implies $\mathcal{N}_{\mathcal{C}}(\mathbf{x}) = \{\mathbf{0}\}$.
2. $\mathbf{x} \in \text{bdy}\{\mathcal{C}\}$ implies $\mathcal{N}_{\mathcal{C}}(\mathbf{x})$ contains nonzero elements.

Proposition 4.1.4 Take closed subsets $\mathcal{C}_1 \subset \mathbb{R}^m$ and $\mathcal{C}_2 \subset \mathbb{R}^n$, and a point $(\mathbf{x}_1, \mathbf{x}_2) \in \mathcal{C}_1 \times \mathcal{C}_2$. Then

$$\mathcal{N}_{\mathcal{C}_1 \times \mathcal{C}_2}^{\mathbf{P}}(\mathbf{x}_1, \mathbf{x}_2) = \mathcal{N}_{\mathcal{C}_1}^{\mathbf{P}}(\mathbf{x}_1) \times \mathcal{N}_{\mathcal{C}_2}^{\mathbf{P}}(\mathbf{x}_2), \quad (4.7)$$

$$\hat{\mathcal{N}}_{\mathcal{C}_1 \times \mathcal{C}_2}(\mathbf{x}_1, \mathbf{x}_2) = \hat{\mathcal{N}}_{\mathcal{C}_1}(\mathbf{x}_1) \times \hat{\mathcal{N}}_{\mathcal{C}_2}(\mathbf{x}_2), \quad (4.8)$$

$$\mathcal{N}_{\mathcal{C}_1 \times \mathcal{C}_2}(\mathbf{x}_1, \mathbf{x}_2) = \mathcal{N}_{\mathcal{C}_1}(\mathbf{x}_1) \times \mathcal{N}_{\mathcal{C}_2}(\mathbf{x}_2). \quad (4.9)$$

Proposition 4.1.5 Take a closed convex set $\mathcal{C} \subset \mathbb{R}^k$ and a point $\bar{\mathbf{x}} \in \mathcal{C}$. Then

$$\hat{\mathcal{N}}_{\mathcal{C}}(\bar{\mathbf{x}}) = \mathcal{N}_{\mathcal{C}}^{\mathbf{P}}(\bar{\mathbf{x}}) = \mathcal{N}_{\mathcal{C}}(\bar{\mathbf{x}}) = \{\boldsymbol{\xi} \mid \langle \boldsymbol{\xi}, \mathbf{x} - \bar{\mathbf{x}} \rangle \leq 0, \quad \forall \mathbf{x} \in \mathcal{C}\}. \quad (4.10)$$

Definition 4.1.6 The Proximal Subgradient Given an extended-valued, lower semi-continuous function $\mathbf{f} : \mathbb{R}^k \rightarrow \mathbb{R} \cup \infty$ and a point $\bar{\mathbf{x}} \in \text{dom } \{\mathbf{f}\}$. A vector $\boldsymbol{\eta} \in \mathbb{R}^k$ is said to be a proximal subgradient of $\mathbf{f}$ at $\bar{\mathbf{x}}$ if there exist $\epsilon > 0$ and $M \geq 0$ such that

$$\langle \boldsymbol{\eta}, \mathbf{x} - \bar{\mathbf{x}} \rangle \leq M \|\mathbf{x} - \bar{\mathbf{x}}\|^2, \forall \mathbf{x} \text{ that satisfy } \|\mathbf{x} - \bar{\mathbf{x}}\| \leq \epsilon. \quad (4.11)$$

Definition 4.1.7 The Proximal Subdifferential The set of all proximal subgradients of $\mathbf{f}$ at $\bar{\mathbf{x}}$ is called the proximal subdifferential of $\mathbf{f}$ at $\bar{\mathbf{x}}$ and is denoted by $\partial^P \mathbf{f}(\bar{\mathbf{x}})$.

Definition 4.1.8 The Limiting Subdifferential Given $\mathbf{f} : \mathbb{R}^k \rightarrow \mathbb{R} \cup \infty$ and a point $\bar{\mathbf{x}} \in \text{dom } \{\mathbf{f}\}$. A vector $\boldsymbol{\eta} \in \mathbb{R}^k$ is said to be a limiting subgradient of $\mathbf{f}$ at $\bar{\mathbf{x}}$ if there exist sequences such that $\mathbf{x}_i \xrightarrow{f} \bar{\mathbf{x}}$ and $\boldsymbol{\eta}_i \rightarrow \boldsymbol{\eta}$ such that

$$\boldsymbol{\eta}_i \in \partial^P \mathbf{f}(\mathbf{x}_i), \quad \forall i. \quad (4.12)$$

The upper subderivative and the lower subderivative of an extended-valued lower semicontinuous (l.s.c) function is defined as given [85].

Definition 4.1.8 Upper and Lower Subderivatives [85] Let f be any extended-real valued lower semi-continuous function on a linear topological space $\mathcal{E}$, and let $\mathbf{x}$ be any point where f is finite. The upper subderivative of f at $\mathbf{x}$ with respect to $\mathbf{y}$ is defined by:

$$f^\uparrow(\mathbf{x}; \mathbf{y}) = \limsup_{\substack{\mathbf{x}' \xrightarrow{f} \mathbf{x} \\ t \downarrow 0}} \inf_{\mathbf{y}' \rightarrow \mathbf{y}} \frac{f(\mathbf{x}' + t\mathbf{y}') - f(\mathbf{x}')}{t}. \quad (4.13)$$

The lower subderivative of f at $\mathbf{x}$ with respect to $\mathbf{y}$ is defined by:

$$f^\downarrow(\mathbf{x}; \mathbf{y}) = \liminf_{\substack{\mathbf{x}' \xrightarrow{f} \mathbf{x} \\ t \downarrow 0}} \sup_{\mathbf{y}' \rightarrow \mathbf{y}} \frac{f(\mathbf{x}' + t\mathbf{y}') - f(\mathbf{x}')}{t}, \quad (4.14)$$

where

$$\mathbf{x}' \xrightarrow{f} \mathbf{x} \Leftrightarrow \mathbf{x}' \Rightarrow \mathbf{x} \quad \wedge \quad f(\mathbf{x}') \Rightarrow f(\mathbf{x}).$$

Theorem 4.1.1 (Theorem (4) in [85]) Let f be any extended-real valued function on a linear topological space $\mathcal{E}$, and let $\mathbf{x}$ be any point where f is finite. Then the "upper" subdifferential $\partial f(\mathbf{x})$ is a weak*-closed convex subset of $\mathcal{E}^*$ and

$$\partial f(\mathbf{x}) = \left\{ \mathbf{z} \in \mathcal{E}^* \mid (\mathbf{z}, -1) \in \mathcal{N}_{\text{epi } f}(\mathbf{x}, f(\mathbf{x})) \right\}. \quad (4.15)$$

If $f^\uparrow(\mathbf{x}; \mathbf{0}) = -\infty$, then $\partial f(\mathbf{x})$ is empty, but otherwise $\partial f(\mathbf{x})$ is nonempty and

$$f^\uparrow(\mathbf{x}; \mathbf{y}) = \sup \{ \langle \mathbf{y}, \mathbf{z} \rangle \mid \mathbf{z} \in \partial f(\mathbf{x}), \quad \forall \mathbf{y} \in \mathcal{E} \}. \quad (4.16)$$

Analogously, the "lower" subdifferential $\tilde{\partial}f(\mathbf{x})$ is a weak*-closed convex subset of $\mathcal{E}^*$ and

$$\tilde{\partial}f(\mathbf{x}) = \left\{ \mathbf{z} \in \mathcal{E}^* \mid (\mathbf{z}, -1) \in \mathcal{N}_{\text{hypo}_f}(\mathbf{x}, f(\mathbf{x})) \right\}. \quad (4.17)$$

If $f^\downarrow(\mathbf{x}; \mathbf{0}) = \infty$, then $\tilde{\partial}f(\mathbf{x})$ is empty, but otherwise $\tilde{\partial}f(\mathbf{x})$ is nonempty and

$$f^\downarrow(\mathbf{x}; \mathbf{y}) = \inf \left\{ \langle \mathbf{y}, \mathbf{z} \rangle \mid \mathbf{z} \in \tilde{\partial}f(\mathbf{x}), \quad \forall \mathbf{y} \in \mathcal{E} \right\}. \quad (4.18)$$

4.2.1 Regularity of Sets

In order to establish the relationship between the geometric concepts defined above and the previously known ones in smooth contexts, a notion of regularity for sets is introduced. The contingent cone $\mathcal{K}_{\mathcal{C}}(\mathbf{x})$ is the set of tangents to a set $\mathcal{C}$ at a point $\mathbf{x}$. A vector $\mathbf{v}$ in $\mathcal{X}$ belongs to $\mathcal{K}_{\mathcal{C}}(\mathbf{x})$ iff, for all $\epsilon > 0$, there exists t in $(0, \epsilon)$ and a point $\mathbf{w}$ in $\mathbf{v} + \epsilon \mathbb{B}$ such that $\mathbf{x} + t\mathbf{w} \in \mathcal{C}$. $\mathcal{T}_{\mathcal{C}}(\mathbf{x})$ is always contained in $\mathcal{K}_{\mathcal{C}}(\mathbf{x})$, which is not always convex.

Definition 4.1.9 The set $\mathcal{C}$ is regular at $\mathbf{x}$ provided $\mathcal{T}_{\mathcal{C}}(\mathbf{x}) = \mathcal{K}_{\mathcal{C}}(\mathbf{x})$. This property implies that $\mathcal{K}_{\mathcal{C}}(\mathbf{x})$ is convex if the set $\mathcal{C}$ is tangentially regular.

Definition 4.1.10 Subdifferential Regularity function f is called subdifferentially regular $\mathbf{x}$ if f is finite at $\mathbf{x}$ and

$$\liminf_{\substack{\mathbf{y}' \rightarrow \mathbf{y} \\ t \downarrow 0}} \frac{f(\mathbf{x} + t\mathbf{y}') - f(\mathbf{x})}{t} = f^\uparrow(\mathbf{x}; \mathbf{y}), \quad \forall \mathbf{y}. \quad (4.19)$$

Theorem 4.1.2 (Rockafellar) Each of the following implies that f is directionally Lipschitzian at $\mathbf{x}$ (a point where f is finite):

1. f is Lipschitzian on a neighborhood of $\mathbf{x}$;
2. f is convex and bounded above on a neighborhood of some point (not necessarily $\mathbf{x}$ itself);
3. f is concave and bounded below on a neighborhood of some point (not necessarily $\mathbf{x}$ itself);
4. f is non-decreasing with respect to the partial ordering of $\mathcal{E}$ induced by some closed convex cone $\mathcal{K}$ with non-empty interior;
5. f is the indicator of a set $\mathcal{C}$ that is epi-Lipschitzian at $\mathbf{x}$;
6. $\mathcal{E} = \mathbb{R}^n$, f is lower semicontinuous on a neighbourhood of $\mathbf{x}$, and the cone $\{\mathbf{y} \mid f^\uparrow(\mathbf{x}; \mathbf{y})\}$ is not included in any subspace of lower dimension;

7. $\mathcal{E} = \mathbb{R}^n$, f is lower semicontinuous on a neighbourhood of $\mathbf{x}$, and $\partial f(\mathbf{x})$ is non-empty and does not include an entire line.

Proposition 4.1.6 Suppose $\mathcal{C}$ is a smooth manifold around $\mathbf{x}$ in the sense that

$$\mathcal{C} = \{\mathbf{x} \mid g_j(\mathbf{x}) = 0, \text{ for } j = 1, \dots, r\}, \quad (4.20)$$

where the functions g_j are continuously differentiable around $\mathbf{y}$ and the gradients $\nabla g_j(\mathbf{x})$, $j = 1, \dots, r$ are linearly independent. Then $\mathcal{K}_{\mathcal{C}}(\mathbf{x})$ is convex, and in fact

$$\mathcal{K}_{\mathcal{C}}(\mathbf{x}) = \{\mathbf{y} \mid \langle \mathbf{y}, \nabla g_j(\mathbf{x}) \rangle = 0, \text{ for } j = 1, \dots, r\}. \quad (4.21)$$

Theorem 4.1.3 Let f_1 and f_2 be extended-real-valued functions on $\mathcal{E}$ that are finite at $\mathbf{x}$. Suppose that f_2 is directionally Lipschitzian at $\mathbf{x}$ and

$$\{\mathbf{y} \mid f_1^\uparrow(\mathbf{x}; \mathbf{y}) < \infty\} \cap \text{int}\{\mathbf{y} \mid f_2^\uparrow(\mathbf{x}; \mathbf{y}) < \infty\} \neq \emptyset \quad (4.22)$$

Then

$$(f_1 + f_2)^\uparrow(\mathbf{x}; \mathbf{y}) \leq f_1^\uparrow(\mathbf{x}; \mathbf{y}) + f_2^\uparrow(\mathbf{x}; \mathbf{y}), \quad \forall \mathbf{y} \quad (4.23)$$

$$\partial(f_1 + f_2)(\mathbf{x}) \subset \partial f_1(\mathbf{x}) + \partial f_2(\mathbf{x}) \quad (4.24)$$

Equality holds in 4.24 if f_1 and f_2 are also subdifferentially regular. It also holds in 4.23 if in addition $f_1^\uparrow(\mathbf{x}; \mathbf{y})$ and $f_2^\uparrow(\mathbf{x}; \mathbf{y})$ are not $-\infty$ (i.e. $\partial f_1(\mathbf{x})$ and $\partial f_2(\mathbf{x})$ are nonempty), and in that event $f_1 + f_2$ is likewise subdifferentially regular.

Corollary 4.1.1 Suppose f_1 is convex and finite at $\mathbf{x}$ and f_2 is Lipschitzian on a neighbourhood of $\mathbf{x}$. Then

$$\partial(f_1 + f_2)(\mathbf{x}) \subset \partial f_1(\mathbf{x}) + \partial f_2(\mathbf{x}), \quad (4.25)$$

and there is equality if f_1 and f_2 are also subdifferentially regular at $\mathbf{x}$.

Corollary 4.1.2 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be subsets of $\mathcal{E}$, and let $\mathbf{x} \in \mathcal{C}_1 \cap \mathcal{C}_2$. Suppose that

$$\mathcal{T}_{\mathcal{C}_1}(\mathbf{x}) \cap \mathcal{T}_{\mathcal{C}_2}(\mathbf{x}) \neq \emptyset, \quad (4.26)$$

and that $\mathcal{C}_2$ is epi-Lipschitzian at $\mathbf{x}$. Then

$$\mathcal{T}_{\mathcal{C}_1 \cap \mathcal{C}_2}(\mathbf{x}) \supset \mathcal{T}_{\mathcal{C}_1}(\mathbf{x}) \cap \mathcal{T}_{\mathcal{C}_2}(\mathbf{x}) \quad (4.27)$$

$$\mathcal{N}_{\mathcal{C}_1 \cap \mathcal{C}_2}(\mathbf{x}) \subset \mathcal{N}_{\mathcal{C}_1}(\mathbf{x}) + \mathcal{N}_{\mathcal{C}_2}(\mathbf{x}) \quad (4.28)$$

where the set on the right in (4.28) is weak*-closed. Equality holds in (4.27) if $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangentially regular at $\mathbf{x}$, and then $\mathcal{C}_1 \cap \mathcal{C}_2$ is likewise tangentially regular.

4.3 The Generalised Problem of Bolza in Impulsive Control Form for Rigidbody Lagrangian Systems

Consider a problem in Bolza form (*GPB*), in which the objective is to choose absolutely continuous arcs $\mathbf{q} \in \mathcal{AC}$ and $\mathbf{u} \in \mathcal{AC}$ in order to minimise problem P given by:

$$P : \quad J(\mathbf{q}, \mathbf{u}) = l(\mathbf{q}(a), \mathbf{u}(a), \mathbf{q}(b), \mathbf{u}(b)) + \int_a^b L(t, \mathbf{q}(t), \mathbf{u}(t), \dot{\mathbf{u}}(t)) dt, \quad (4.29)$$

where the function $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is $\mathcal{L} \times \mathcal{B}$ measurable. Here $\mathcal{L} \times \mathcal{B}$ denotes the σ -algebra of subsets of $[a, b] \times \mathbb{R}^n$ generated by product sets $\mathcal{M} \times \mathcal{N}$, where $\mathcal{M}$ is a Lebesgue measurable subset of $[a, b]$ and $\mathcal{N}$ is a Borel subset of $\mathbb{R}^{3n}$. For each $t \in [a, b]$, the function l and L are lower semi-continuous on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, respectively, with values in $\mathbb{R} \cup \{+\infty\}$. For each $(t, \mathbf{q}, \mathbf{u})$ in $[a, b] \times \mathbb{R}^n \times \mathbb{R}^n$, the function $L(t, \mathbf{q}, \mathbf{u}, \cdot)$ is convex and l represents the endpoint cost. GPB concerns the minimisation of a functional whose form is identical to that in the classical calculus of variations. The endpoint cost l and the integrand L are allowed to take the value $+\infty$, so that a variety of endpoint and differential constraints can be treated. An important class of optimal control problems constrain the derivative of an admissible arc and can be formulated in the form:

$$\min\{l(\mathbf{q}(a), \mathbf{u}(a), \mathbf{q}(b), \mathbf{u}(b)) : \mathbf{M}(\mathbf{q}) \dot{\mathbf{u}}(t) \in \mathcal{F}(t, \mathbf{q}, \mathbf{u}) \quad \text{a.e.} \quad t \in [a, b]\}. \quad (4.30)$$

The problem in (4.30) can be seen as minimising the Bolza functional J over all arcs $\mathbf{q}, \mathbf{u}$. If one identifies the integrand in (4.29) with (4.31):

$$L(t, \mathbf{q}, \mathbf{u}, \mathbf{a}) = \Psi_{\mathcal{F}(t, \mathbf{q}, \mathbf{u})}(\mathbf{a}) = \begin{cases} 0, & \text{if } \mathbf{M}(\mathbf{q}) \mathbf{a} \in \mathcal{F}(t, \mathbf{q}, \mathbf{u}), \\ \text{else } +\infty, \end{cases} \quad (4.31)$$

the general class of optimal control problems defined in (4.30) can be handled as a GPB. It is evident that for some pair of arcs $(\mathbf{q}, \mathbf{u})$ one then has

$$\int_a^b L(t, \mathbf{q}, \mathbf{u}, \dot{\mathbf{u}}) dt = \begin{cases} 0, & \mathbf{M}(\mathbf{q}) \dot{\mathbf{u}}(t) \in \mathcal{F}(t, \mathbf{q}, \mathbf{u}) \quad \text{a.e.} \quad t \in [a, b], \\ +\infty. \end{cases} \quad (4.32)$$

where the differential inclusion $\mathcal{F}$ is defined via the state-control triplet $(\boldsymbol{\tau}, \mathbf{q}, \mathbf{u})$:

$$\begin{aligned} \mathbf{M}(\mathbf{q}) \dot{\mathbf{u}}(t) &\in \mathcal{F}(\mathbf{q}, \mathbf{u}) \\ &:= \{\mathbf{a} \mid \mathbf{M}(\mathbf{q}) \mathbf{a} = \mathbf{h}(\mathbf{q}(t), \mathbf{u}(t)) + \mathbf{B}(\mathbf{q}(t)) \boldsymbol{\tau}(t) \quad \forall \boldsymbol{\tau}(t) \in \mathcal{C}_\tau, \text{ a.e. } t \in [a, b]\}. \end{aligned} \quad (4.33)$$

In order to guarantee the well-behaving of $\mathcal{F}$ and l let following assumptions hold:

Assumptions 4.1. A pair of trajectories of generalised positions $\bar{\mathbf{q}} : [a, b] \rightarrow \mathbb{R}^n$ and velocities $\bar{\mathbf{u}} : [a, b] \rightarrow \mathbb{R}^n$ is given. On some relatively open subset $\Omega \subseteq [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ containing the graph of $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$, the following statements hold:

1. The multifunction $\mathcal{F}$ is $\mathcal{L} \times \mathcal{B}$ measurable on Ω . For each $(t, \mathbf{q}, \mathbf{u})$ in Ω , the set $\mathcal{F}(t, \mathbf{q}, \mathbf{u})$ is nonempty, compact and convex.
2. There are nonnegative integrable functions $k_{\mathbf{q}}(t)$, $k_{\mathbf{u}}(t)$ and $\Phi(t)$ on $[a, b]$ such that
 - (a) $\mathcal{F}(t, \mathbf{q}, \mathbf{u}) \subseteq \Phi(t) \mathbb{B}$ for all $\mathbf{q}, \mathbf{u} \in \Omega_T$, almost everywhere, and
 - (b) $\mathcal{F}(t, \mathbf{q}, \mathbf{u}) \subseteq \mathcal{F}(t, \mathbf{q}, \mathbf{u}) + k_{\mathbf{q}}(t)|\mathbf{p} - \mathbf{r}| + k_{\mathbf{u}}(t)|\dot{\mathbf{p}} - \dot{\mathbf{r}}| \mathbb{B}$ for all $(\mathbf{p}, \dot{\mathbf{p}}), (\mathbf{r}, \dot{\mathbf{r}}) \in \Omega_T$, almost everywhere.
3. The endpoint cost function l is lower-semicontinuous on $\Omega_a \times \Omega_b$.

Here Ω_T is given by $\Omega_T = \{(\mathbf{q}, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid (t, \mathbf{q}, \mathbf{u}) \in \Omega, \forall t \in [a, b] \setminus \{t_i\} \in \mathcal{I}_T\}$ and $\mathbb{B}$ is the unit open ball. It is assumed that conditions of assumptions 4.1 are fulfilled for the Lebesgue-measurable part of the Lagrangian dynamics in the "almost everywhere" sense. By the way Ω_T is defined, the instants of the discontinuity are excluded.

Having set the stage, the necessary conditions for the impulsive optimal control problem of finite-dimensional Lagrangian systems is formally derived by considering a problem in GPB, in which the objective is to choose arcs $\mathbf{q} \in \mathcal{AC}$ and $\mathbf{u} \in \mathcal{LBV}$ on every interval $(t_i^+, t_{i+1}^-) \in \mathcal{I}_T$ and transition location triplets $\{\mathbf{q}(t_{i+1}), \mathbf{u}(t_{i+1}^-), \mathbf{u}(t_i^+)\}$, transition times t_i and final time t_f , $\forall t_i \in \mathcal{I}_T$ in order to minimise P_{Tot} :

$$J(\mathbf{q}(t), \mathbf{u}(t), \{\mathbf{q}(t_i), \mathbf{u}(t_i^-), \mathbf{u}(t_i^+), t_i^-, t_i^+\}, t_f) = \quad (4.34)$$

$$\sum_{i=1}^N l(\mathbf{q}(t_{i+1}^-), \mathbf{u}(t_{i+1}^-), \mathbf{q}(t_i^+), \mathbf{u}(t_i^+)) + \int_{t_i^+}^{t_{i+1}^-} L_i(s_i, \mathbf{q}(s_i), \mathbf{u}(s_i), \dot{\mathbf{u}}(s_i)) ds_i.$$

The theory at hand treats optimal solutions as solutions of multi-point boundary value problems (MBVP) with discontinuous transitions in the state. In this setting, the prespecification of the mode sequence and number of intervals must be given in advance. The overall problem as stated in (4.34) is seen as the union of several problems in the generalised Bolza form. Here it is assumed that the control horizon is composed of n different phases, which are separated from each other by $N - 1$ possibly discontinuous transitions in the generalised velocities. The importance of the transition process becomes clear if one considers the fact that at pre-transition and post-transition states the values of several functions may differ due to discontinuities. A transition process is common to the pre-transition configuration and post-transition configuration. Each problem P_i with a unique mechanical configuration is defined on a closed time domain $\text{dom}(P_i)$ with variable boundary which is partitioned as follows:

$$\text{dom}(P_i) = \{t_i^-, t_i^+\} \bigcup (t_i^+, t_{i+1}^-) \bigcup \{t_{i+1}^-, t_{i+1}^+\}. \quad (4.35)$$

The boundary of the domain $\text{dom}(P_i)$ is given by:

$$\text{bdy dom}(P_i) = \{t_i^-, t_i^+\} \bigcup \{t_{i+1}^-, t_{i+1}^+\}. \quad (4.36)$$

The interior of the domain is given by:

$$\text{int dom}(P_i) = (t_i^+, t_{i+1}^-). \quad (4.37)$$

The domain of the overall problem P_{Tot} is given by the union:

$$\text{dom}(P_{\text{Tot}}) = \bigcup_{\forall i} \text{dom}(P_i). \quad (4.38)$$

However, the domains of successive problems P_i and P_{i+1} are not disjoint:

$$\text{dom}(P_i) \cap \text{dom}(P_{i+1}) = \text{bdy dom}(P_i) \cap \text{bdy dom}(P_{i+1}) = \{t_{i+1}^-, t_{i+1}^+\}, \quad (4.39)$$

The set $\text{bdy dom}(P_i) \cap \text{bdy dom}(P_{i+1}) = \{t_{i+1}^-, t_{i+1}^+\}$ is the support of the transition process and is Lebesgue-negligible. The extended-valued integrand may differ on each interval based on the structure of the equations of motion. The difference in structure may arise due to change in parameters (i.e. mass, inertia) or degrees of freedom. In [112] by Yunt a projection approach is presented in case, the mechanical configurations in successive intervals differ based on change in the number of degrees of freedom.

4.4 Statement of the Optimal Control Problem

The impulsive optimal control of multibody Lagrangian systems is considered, for which the transition times $t_i \in \mathcal{I}_T$, final time t_f and transition location triplets $\{\mathbf{q}(t_i), \mathbf{u}(t_i^+), \mathbf{u}(t_i^-)\}$ are free. The goal function is to minimise the functional $g(\mathbf{q}, \mathbf{u}, \boldsymbol{\tau})$. The differential inclusion of $\{\mathbf{q}(t), \mathbf{u}(t), \dot{\mathbf{u}}(t), \boldsymbol{\tau}(t)\}$ that fulfill the Lebesgue measurable part of the dynamics in every time-interval (t_i^+, t_{i+1}^-) is denoted by $\mathcal{F}_i$:

$$\mathbf{M}(\mathbf{q}(t))\dot{\mathbf{u}}(t) \in \mathcal{F}_i(\mathbf{q}(t), \mathbf{u}(t)), \quad t \in (t_i^+, t_{i+1}^-) \quad \text{a.e.} \quad (4.40)$$

The measurable controls $\boldsymbol{\tau}$ is constrained to a bounded closed polytopic convex set $\mathcal{C}_\tau$. The set $\mathcal{C}_{\mathbf{I}_i}^+$ denotes the set of $\{\mathbf{q}(t_i^+), \mathbf{u}(t_i^+), \mathbf{u}(t_i^-), \boldsymbol{\zeta}_i^+, \boldsymbol{\zeta}_i^-\}$ that fulfill the post-transition impact equation:

$$\mathbf{M}(\mathbf{q}(t_i^+))(\mathbf{u}(t_i^+) - \mathbf{u}(t_i^-)) - \mathbf{B}_i(\mathbf{q}(t_i^+))(\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) = \mathbf{0}, \quad \forall t_i \in \mathcal{I}_T. \quad (4.41)$$

The set $\mathcal{C}_{\mathbf{I}_i}^-$ denotes the set of $\{\mathbf{q}(t_i^-), \mathbf{u}(t_i^+), \mathbf{q}(t_i^-), \boldsymbol{\zeta}_i^+, \boldsymbol{\zeta}_i^-\}$ that fulfill the pre-transition impact equation:

$$\mathbf{M}(\mathbf{q}(t_i^-))(\mathbf{u}(t_i^+) - \mathbf{u}(t_i^-)) - \mathbf{B}_i(\mathbf{q}(t_i^-))(\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) = \mathbf{0}, \quad \forall t_i \in \mathcal{I}_T. \quad (4.42)$$

The equations (4.41) and (4.42) represent smooth manifolds and are smoothly differentiable in their arguments. Analogously, let $\mathcal{C}_{\mathbf{T}_i}^+$ denote the set defined by the equality

$$\mathbf{p}_i^+(\mathbf{q}(t_i^+), \mathbf{u}(t_i^+), \mathbf{u}(t_i^-)) = \mathbf{0}, \quad (4.43)$$

and $\mathcal{C}_{T_i}^-$ denote the set defined by the equality

$$\mathbf{p}_i^-(\mathbf{q}(t_i^-), \mathbf{u}(t_i^+), \mathbf{u}(t_i^-)) = \mathbf{0}, \quad (4.44)$$

that arise from the pre-transition and post-transition constitutive impact laws. Both $\mathbf{p}_i^-$ and $\mathbf{p}_i^+$ are at least C^1 in their arguments. The end state is to be in a convex set $\mathcal{C}_f(\mathbf{q}(t_f), \mathbf{u}(t_f))$. By the absolute continuity of the generalised positions, the relations:

$$\mathcal{C}_{T_i} = \mathcal{C}_{T_i}^+ \cup \mathcal{C}_{T_i}^- = \mathcal{C}_{T_i}^+ \cap \mathcal{C}_{T_i}^-, \quad \forall t_i \in \mathcal{I}_T, \quad (4.45)$$

$$\mathcal{C}_{I_i} = \mathcal{C}_{I_i}^+ \cup \mathcal{C}_{I_i}^- = \mathcal{C}_{I_i}^+ \cap \mathcal{C}_{I_i}^-, \quad \forall t_i \in \mathcal{I}_T, \quad (4.46)$$

are tractable. In its full glory the impulsive optimal control problem is stated as:

$$\min_{\{t_i\}, t_f, \boldsymbol{\tau}, \{\boldsymbol{\zeta}_i^+, \boldsymbol{\zeta}_i^-\}} J, \quad (4.47)$$

where J is given by:

$$J = l_0 + \sum_{i=1}^N l_i + \sum_{i=1}^N \int_{t_i^+}^{t_{i+1}^-} L_i(\mathbf{q}(s), \mathbf{u}(s), \dot{\mathbf{u}}(s)) ds. \quad (4.48)$$

The costs associated with boundary terms and the integrand are composed in the following manner:

$$l_i = \Psi_{\mathcal{C}_{T_i}^+}(\mathbf{q}(t_i^+), \mathbf{u}(t_i^+), \mathbf{u}(t_i^-)) + \Psi_{\mathcal{C}_{T_i}^-}(\mathbf{q}(t_i^-), \mathbf{u}(t_i^+), \mathbf{u}(t_i^-)) \quad (4.49)$$

$$+ \Psi_{\mathcal{C}_{I_i}^-}(\mathbf{q}(t_i^-), \mathbf{u}(t_i^+), \mathbf{u}(t_i^-), \boldsymbol{\zeta}_i^+, \boldsymbol{\zeta}_i^-) + \Psi_{\mathcal{C}_{I_i}^+}(\mathbf{q}(t_i^+), \mathbf{u}(t_i^+), \mathbf{u}(t_i^-), \boldsymbol{\zeta}_i^+, \boldsymbol{\zeta}_i^-),$$

$$l_0 = \Psi_{\mathcal{C}_f}(\mathbf{q}(t_f), \mathbf{u}(t_f)), \quad (4.50)$$

$$L_i = \lambda(t) g(\mathbf{q}(t), \mathbf{u}(t), \boldsymbol{\tau}) + \Psi_{\text{Gr } \mathcal{F}_i^+(\mathbf{q}(t^+), \mathbf{u}(t^+))}. \quad (4.51)$$

The necessary conditions are derived by making use of following assumptions on the general problem:

Assumptions 4.2

1. the dual states $\boldsymbol{\nu}$ is assumed left-continuous locally bounded variation functions ($\mathcal{LCLBV}$), and the generalised velocities $\mathbf{u}$ of the Lagrangian system is assumed right-continuous locally bounded variation functions ($\mathcal{RCLBV}$), whereas the generalised positions are in class $\mathcal{AC}$.
2. The mode sequence and number of intervals for the MBVP constitute a feasible hybrid trajectory.
3. The set $\mathcal{C}_{I_i}^+ \cap \mathcal{C}_{T_i}^+$ is closed and nonempty.

4. The set $\mathcal{C}_{I_i}^- \cap \mathcal{C}_{T_i}^-$ is closed and nonempty.
5. The goal functional $g(\mathbf{q}, \mathbf{u}, \boldsymbol{\tau})$ is convex for all $t \in \Omega_t$ and $t_i \in \mathcal{I}_T$.
6. The limiting partial subdifferential $\partial_{\mathbf{u}} g(\mathbf{q}, \mathbf{u}, \boldsymbol{\tau})$ is bounded for all $t \in \Omega_t$ and $t_i \in \mathcal{I}_T$.
7. Each $L_i : (t_i^+, t_{i+1}^-) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a Lebesgue normal integrand.
8. Each $L_i(\mathbf{q}(s), \mathbf{u}(s), \cdot)$ is convex for each $(\mathbf{q}(s), \mathbf{u}(s))$.
9. Each $l_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

Assumption 4.2.8 is equivalent to the epigraph of each L_i

$$\text{epi } L_i(\cdot, \cdot, \cdot) = \{(\mathbf{q}(s), \mathbf{u}(s), \dot{\mathbf{u}}(s), u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \mid u \geq L_i(\mathbf{q}(s), \mathbf{u}(s), \dot{\mathbf{u}}(s))\} \quad (4.52)$$

being closed and depending Lebesgue measurably on t , in the sense that for each closed $\mathcal{V} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, the set

$$\left\{t \in (t_i^+, t_{i+1}^-) \mid \mathcal{V} \cap \text{epi } L_i(\cdot, \cdot, \cdot) \neq \emptyset\right\} \quad (4.53)$$

is Lebesgue measurable. Normality implies that $L_i(\cdot, \cdot, \cdot)$ is Lebesgue measurable in t whenever $(\mathbf{q}(t), \mathbf{u}(t), \dot{\mathbf{u}}(t))$ is and that each $L_i(\mathbf{q}(t), \mathbf{u}(t), \dot{\mathbf{u}}(t))$ is lower semicontinuous for each $t \in (t_i^+, t_{i+1}^-)$. These results are given in [83] by Rockafellar.

The differentiability properties of the transition sets given by the impact equations and the constitutive impact laws are sufficient to render each l_i defined by (4.49) in the finite-dimensional case subdifferentially regular. As stated in [84] by Rockafellar if an extended-valued function f is the indicator function of a set $\mathcal{C}$ that is tangentially regular then it is subdifferentially regular. In this case, the set of which l_i is the indicator function is given by:

$$\mathcal{C} = (\mathcal{C}_{I_i}^- \cup \mathcal{C}_{I_i}^+) \cap (\mathcal{C}_{T_i}^- \cup \mathcal{C}_{T_i}^+) = (\mathcal{C}_{I_i}^- \cap \mathcal{C}_{I_i}^+) \cap (\mathcal{C}_{T_i}^- \cap \mathcal{C}_{T_i}^+) \quad (4.54)$$

and is tangentially regular by the differentiability properties of the underlying equations. The set defined in (4.54) is nonempty and closed by assumptions 4.2.4 and 4.2.5 and properties (4.45) and (4.46). The closedness of this set is equivalent to the lower semi-continuity as required by assumption 4.2.10. Assumptions 4.2.4 and 4.2.5 are sufficient to render each l_i defined by (4.49) in the finite-dimensional case directionally Lipschitzian by theorem 1.e in [84] by Rockafellar (Theorem 4.1.2).

The partial directional derivatives with respect to $\hat{\mathbf{q}}(t^+)$, $\hat{\mathbf{u}}(t^+)$, $\hat{\dot{\mathbf{u}}}(t^+)$, $\hat{\boldsymbol{\tau}}(t^+)$, $\hat{t}_f$, $\hat{t}_i^+$, $\hat{t}_i^-$, $\hat{\mathbf{u}}(t_i^+)$, $\hat{\mathbf{u}}(t_i^-)$, $\hat{\mathbf{u}}(t_f)$, $\hat{\mathbf{q}}(t_i^+)$, $\hat{\mathbf{q}}(t_i^-)$, $\hat{\mathbf{q}}(t_f)$, $\hat{\boldsymbol{\zeta}}_i^+$, $\hat{\boldsymbol{\zeta}}_i^-$ need to be evaluated. Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{q}}(t^+)$ reveals:

$$\sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\mathbf{q}}(t^+)) + \Psi_{\mathcal{C}_\tau}^\uparrow(\cdot, \hat{\mathbf{q}}(t^+)) dt. \quad (4.55)$$

The directional derivatives of the indicator function of the sets $\mathcal{F}_i^+$ and $\mathcal{C}_\tau$ in the direction $\hat{\mathbf{q}}(t^+)$ are by proposition 4.1.6 given as:

$$\Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\mathbf{q}}(t^+)) = \boldsymbol{\nu}(t^-) \nabla_{\mathbf{q}} S_i^+ \hat{\mathbf{q}}(t^+), \quad t \in (0, t_i^-), \quad \forall t \in \text{int dom}(P_i), \quad (4.56)$$

$$\Psi_{\mathcal{C}_\tau}^\uparrow(\cdot, \hat{\mathbf{q}}(t^+)) = 0, \quad \forall t \in \text{int dom}(P_i). \quad (4.57)$$

The partial directional derivative of the value function J in direction $\hat{\mathbf{q}}(t^+)$ is by proposition 4.1.6 given as:

$$\sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \boldsymbol{\nu}(t^-) \nabla_{\mathbf{q}} S_i^+ \hat{\mathbf{q}}(t^+) dt. \quad (4.58)$$

where S_i^+ is defined as:

$$S_i^+ = \mathbf{M}(\mathbf{q}) \dot{\mathbf{u}}^+ - \mathbf{h}_i(\mathbf{q}, \mathbf{u}^+) - \mathbf{B}_i(\mathbf{q}) \boldsymbol{\tau}^+, \quad (4.59)$$

The time-dependent $\boldsymbol{\nu}(t^-)$ vector is an element of the dual space. Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{u}}(t^+)$ reveals:

$$\sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t^+)) + \Psi_{\mathcal{C}_\tau}^\uparrow(\cdot, \hat{\mathbf{u}}(t^+)) dt, \quad (4.60)$$

where the directional derivatives of the indicator function of the sets $\mathcal{F}_i^+$ and $\mathcal{C}_\tau$ in the direction $\hat{\mathbf{u}}(t^+)$ are:

$$\Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t^+)) = \boldsymbol{\nu}(t^-) \nabla_{\mathbf{u}} \mathbf{S}_i^+ \hat{\mathbf{u}}(t^+), \quad t \in (0, t_i^-), \quad \forall t \in \text{int dom}(P_i), \quad (4.61)$$

$$\Psi_{\mathcal{C}_\tau}^\uparrow(\cdot, \hat{\mathbf{u}}(t^+)) = 0, \quad \forall t \in \text{int dom}(P_i). \quad (4.62)$$

Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{u}}(t^+)$

$$J^\uparrow(\cdot, \hat{\mathbf{u}}(t^+)) = \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t^+)) dt, \quad (4.63)$$

where the directional derivative $\Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t^+))$ is equivalent to:

$$\Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t^+)) = \boldsymbol{\nu}(t^-) \mathbf{M}(\mathbf{q}) \hat{\mathbf{u}}(t^+), \quad \forall t \in \text{int dom}(P_i). \quad (4.64)$$

Evaluation of the partial directional derivative of J in direction $\hat{\boldsymbol{\tau}}(t^+)$ reveals:

$$\sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\boldsymbol{\tau}}(t^+)) + \Psi_{\mathcal{C}_\tau}^\uparrow(\cdot, \hat{\boldsymbol{\tau}}(t^+)) dt, \quad (4.65)$$

where the directional derivatives of the indicator function of the sets $\mathcal{F}_i^+$ and $\mathcal{C}_\tau$ in the direction $\hat{\tau}(t^+)$ are:

$$\Psi_{\mathcal{F}_i^+}^\uparrow(\cdot, \hat{\tau}(t^+)) = \nu(t^-) \nabla_\tau \mathbf{S}_i^+ \hat{\tau}(t^+), \quad \forall t \in \text{int dom}(P_i). \quad (4.66)$$

$$(4.67)$$

Evaluation of the partial directional derivative of J in direction $\hat{t}_f$ results in:

$$(\lambda(t_f) g(\mathbf{q}(t_f), \mathbf{u}(t_f), \boldsymbol{\tau}(t_f)) + \Psi_{\mathcal{F}_i^+} + \Psi_{\mathcal{C}_\tau}) \hat{t}_f. \quad (4.68)$$

Evaluation of the partial directional derivative of J in direction $\hat{t}_i^+$ results in:

$$(\lambda(t_i^+) g(\mathbf{q}(t_i^+), \mathbf{u}(t_i^+), \boldsymbol{\tau}(t_i^+)) + \Psi_{\mathcal{F}_i^+} + \Psi_{\mathcal{C}_\tau}) \hat{t}_i^+, \quad \forall t_i \in \mathcal{I}_T. \quad (4.69)$$

Evaluation of the partial directional derivative of J in direction $\hat{t}_i^-$ results in:

$$(\lambda(t_i^-) g(\mathbf{q}(t_i^-), \mathbf{u}(t_i^-), \boldsymbol{\tau}(t_i^-)) + \Psi_{\mathcal{C}_\tau}) \hat{t}_i^-, \quad \forall t_i \in \mathcal{I}_T. \quad (4.70)$$

Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{u}}(t_i^+)$ results in:

$$\Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t_i^+)) + \Psi_{\mathcal{C}_{T_i}^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t_i^+)) + \Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\mathbf{u}}(t_i^+)) + \Psi_{\mathcal{C}_{T_i}^-}^\uparrow(\cdot, \hat{\mathbf{u}}(t_i^+)), \quad \forall t_i \in \mathcal{I}_T. \quad (4.71)$$

Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{u}}(t_i^-)$ results in:

$$\Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t_i^-)) + \Psi_{\mathcal{C}_{T_i}^+}^\uparrow(\cdot, \hat{\mathbf{u}}(t_i^-)) + \Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\mathbf{u}}(t_i^-)) + \Psi_{\mathcal{C}_{T_i}^-}^\uparrow(\cdot, \hat{\mathbf{u}}(t_i^-)), \quad \forall t_i \in \mathcal{I}_T. \quad (4.72)$$

Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{u}}(t_f)$ results in:

$$\Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{\mathbf{u}}(t_f)). \quad (4.73)$$

Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{q}}(t_i^+)$ results in:

$$\Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^+)) + \Psi_{\mathcal{C}_{T_i}^+}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^+)), \quad \forall t_i \in \mathcal{I}_T. \quad (4.74)$$

Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{q}}(t_i^-)$ results in:

$$\Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^-)) + \Psi_{\mathcal{C}_{T_i}^-}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^-)), \quad \forall t_i \in \mathcal{I}_T. \quad (4.75)$$

Evaluation of the partial directional derivative of J in direction $\hat{\mathbf{q}}(t_f)$ results in:

$$\Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{\mathbf{q}}(t_f)), \quad \forall t \in \text{int dom}(P_i). \quad (4.76)$$

Evaluation of the partial directional derivative of J in direction $\hat{\boldsymbol{\zeta}}_i^+$ results in:

$$\Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^+) + \Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^+), \quad \forall t_i \in \mathcal{I}_T. \quad (4.77)$$

Evaluation of the partial directional derivative of J in direction $\hat{\boldsymbol{\zeta}}_i^-$ results in:

$$\Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^-) + \Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^-), \quad \forall t_i \in \mathcal{I}_T. \quad (4.78)$$

4.5 Internal Boundary Variations and Discontinuous Transversality Conditions

In order to obtain criteria for the optimality of the transition position, transition pre-, and post-transition generalised velocities, transition time and impulsive control, variations in these entities are considered. Given the lower-semicontinuous extended-valued functional J a variational inequality of the form:

$$J^\uparrow(\cdot, \mathbf{y}(t_i)) + \langle \mathbf{m}(t_i), \mathbf{y}(t_i) \rangle \geq 0, \quad \forall t_i \in \mathcal{I}_T. \quad (4.79)$$

is sought, in order to formulate the necessary conditions under regularity assumptions for which (4.79) becomes the necessary condition. Here the direction $\mathbf{y}$ is given by:

$$\mathbf{y}(t_i) = \begin{bmatrix} \mathbf{u}(t_i^+) \\ \mathbf{u}(t_i^-) \\ \mathbf{q}(t_i) \\ \zeta_i^+ \\ \zeta_i^- \end{bmatrix}. \quad (4.80)$$

The structure of the vector $\mathbf{m}(t_i)$ becomes clear in the course of the derivation. It is assumed that the mechanics as well as the transition conditions are not explicitly dependent on time or transition time. In order to obtain a variational inequality for transition time of the form:

$$J^\uparrow(\cdot, \hat{t}_i) + \langle \mathbf{m}_{t_i}, \hat{t}_i \rangle \geq 0 \quad \forall t_i \in \mathcal{I}_T. \quad (4.81)$$

requires assumptions on the dependence of J on transition time t_i . This dependence is embedded in the concept of internal boundary variations. The evaluation of the total subderivative of the value function in the orthogonal boundary variations reveal the discontinuous transversality conditions for each transition time $t_i \in \mathcal{I}_T$. The set of such orthogonal boundary variations is denoted by $\hat{\mathcal{V}}$. The optimality condition requires that the lower subderivatives of the value functional $J^\downarrow(\cdot; \hat{\psi})$ are all nonnegative with respect to the admissible boundary variations:

$$J^\downarrow(\cdot; \hat{\psi}) \geq 0, \quad \forall \hat{\psi} \in \hat{\mathcal{V}} \text{ and } \hat{\psi} \text{ admissible.} \quad (4.82)$$

The functional J is directionally Lipschitzian in all directions $\hat{\psi} \in \hat{\mathcal{V}}$ as shown in section 4.3. By theorem (6) in [85] this property of J implies the equivalence of the "lower" and "upper" subdifferentials of J :

$$\partial J = \tilde{\partial} J. \quad (4.83)$$

By reverting to the definition of the "upper" and "lower" subdifferential, one notices that the lower and upper subderivatives coincide in the directionally Lipschitzian case:

$$J^\downarrow(\cdot; \hat{\psi}) = J^\uparrow(\cdot; \hat{\psi}), \quad \forall \hat{\psi} \in \hat{\mathcal{V}}. \quad (4.84)$$

The upper subderivatives of the functional J have following structure in the evaluation of the internal boundary variations:

$$J^\uparrow(\cdot; \hat{\psi}_i) = \sum_{j=0}^{N-1} l_j^\uparrow(\cdot; \hat{\psi}_i) + \sum_{j=0}^{N-1} \left(\int_{t_j^+}^{t_{j+1}^-} L_j(\mathbf{q}(s), \mathbf{u}(s), \dot{\mathbf{u}}(s)) ds \right)^\uparrow(\cdot; \hat{\psi}_i), \quad \forall \hat{\psi}_i \in \hat{\mathcal{V}}. \quad (4.85)$$

4.5.1 Internal Boundary Variations

In this subsection the internal boundary variations at the transition from subproblem P_{i-1} to P_i is investigated. In the classical calculus of variations where the final state and final time are free, the variations of the final state are composed of two parts, namely, the part that arises of the variations at a given time and the part arising from variations due to final time. Since the transitions times are assumed to be free, the two-part character of the variations at pre- and post-transition states is considered. By the Lebesgue-Stieltjes integration of the differential measure of the generalised velocities following relation:

$$\int_{\{t_i\}} d\mathbf{u} = \mathbf{u}(t_i^+) - \mathbf{u}(t_i^-) = \boldsymbol{\chi}_i^+ - \boldsymbol{\chi}_i^-, \quad (4.86)$$

is obtained, whereas for the generalised positions one has

$$\int_{\{t_i\}} d\mathbf{q} = \mathbf{q}(t_i^+) - \mathbf{q}(t_i^-) = \mathbf{0}, \quad (4.87)$$

by assumption 4.3.2. Based on the relations (4.86) and (4.87), the variations of the pre-, and post-transition generalised positions and velocities at fixed time $\hat{\mathbf{q}}(t_i^+)$, $\hat{\mathbf{u}}(t_i^+)$, $\hat{\mathbf{q}}(t_i^-)$, $\hat{\mathbf{u}}(t_i^-)$ are brought in relation with the total variations in these entities $\hat{\mathbf{q}}_i^+$, $\hat{\mathbf{u}}_i^+$, $\hat{\mathbf{q}}_i^-$, $\hat{\mathbf{u}}_i^-$ at each $t_i \in \mathcal{I}_T$ by the following affine relations:

$$\hat{\mathbf{q}}(t_i^+) = \hat{\mathbf{q}}_i^+ - \mathbf{u}(t_i^+) \hat{t}_i^+, \quad (4.88)$$

$$\hat{\mathbf{q}}(t_i^-) = \hat{\mathbf{q}}_i^- - \mathbf{u}(t_i^-) \hat{t}_i^-, \quad (4.89)$$

$$\hat{\mathbf{u}}(t_i^+) = \hat{\mathbf{u}}_i^+ - \dot{\mathbf{u}}(t_i^+) \hat{t}_i^+ - \hat{\boldsymbol{\chi}}_i^+, \quad (4.90)$$

$$\hat{\mathbf{u}}(t_i^-) = \hat{\mathbf{u}}_i^- - \dot{\mathbf{u}}(t_i^-) \hat{t}_i^- - \hat{\boldsymbol{\chi}}_i^-. \quad (4.91)$$

By making use of the affine relations given in equations (4.88) to (4.91) the boundary variations are decomposed into orthogonal independent variations in $\hat{t}_i^-$, $\hat{t}_i^+$, $\hat{\mathbf{u}}_i^+$, $\hat{\mathbf{u}}_i^-$, $\hat{\mathbf{q}}_i^+$, $\hat{\mathbf{q}}_i^-$, $\hat{\boldsymbol{\zeta}}_i^-$, $\hat{\boldsymbol{\zeta}}_i^+$ at each transition instant. Thus the internal boundary variations at each transition time are given in the finite-dimensional set $\hat{\mathcal{V}}$:

$$\begin{aligned} \hat{\mathcal{V}} &= \left\{ \hat{t}_i^-, \hat{t}_i^+, \hat{\mathbf{u}}_i^+, \hat{\mathbf{u}}_i^-, \hat{\mathbf{q}}_i^+, \hat{\mathbf{q}}_i^-, \hat{\boldsymbol{\zeta}}_i^+, \hat{\boldsymbol{\zeta}}_i^-, \hat{\boldsymbol{\chi}}_i^-, \hat{\boldsymbol{\chi}}_i^+ \right\} \\ &\subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^n. \end{aligned} \quad (4.92)$$

The assumptions during a possibly impactive transition are given as follows:

Assumptions 4.3

1. The transitions may be impactively.
2. The generalised position remain unchanged during transition.
3. The impulsive control action acts on the system at a time instant t_i which is Lebesgue-negligible and of which there are countably many.
4. At a possibly impactive transition, the pre-transition controller configuration is assumed to be effective.
5. There are no transitions at initial time t_0 and final time t_f without loss of generality.

The above stated assumptions are converted into requirements to the variations at the internal boundaries.

4.5.2 Discontinuous Transversality Conditions

The subderivatives of the integral part of condition (4.85) is considered first:

$$\left(\int_{t_i^+}^{t_{i+1}^-} L_i(\mathbf{q}(s), \mathbf{u}(s), \dot{\mathbf{u}}(s)) ds + \int_{t_{i-1}^+}^{t_i^-} L_i(\mathbf{q}(s), \mathbf{u}(s), \dot{\mathbf{u}}(s)) ds \right)^\uparrow (\cdot; \hat{\psi}_i), \quad \forall \hat{\psi}_i \in \hat{\mathcal{V}}. \quad (4.93)$$

In order to access the boundary variations and eliminate the variations with respect to $\hat{\mathbf{u}}(t^+)$ and $\hat{\mathbf{u}}(t^+)$ the Lemma of Du Bois-Reymond is used. In the process of eliminating the variation with respect to $\hat{\mathbf{u}}(t^+)$ on every open interval (t_i^+, t_{i+1}^-) such that $\{t_i, t_{i+1}\} \in \mathcal{I}_T$ the Lemma of Du Bois-Reymond is applied twice successively:

$$\int_{t_i^+}^{t_i^-} \boldsymbol{\nu}(t^-) \mathbf{M}(\mathbf{q}(t^+)) \hat{\mathbf{u}}(t^+) dt = \boldsymbol{\nu}(t^-) \mathbf{M}(\mathbf{q}(t^+)) \hat{\mathbf{u}}(t^+) \Big|_{t_i^+}^{t_i^-} \quad (4.94)$$

$$\begin{aligned} & - \int_{t_i^+}^{t_i^-} \left(\dot{\boldsymbol{\nu}}(t^-) \mathbf{M}(\mathbf{q}(t^+)) + \boldsymbol{\nu}(t^-) \dot{\mathbf{M}}(\mathbf{q}(t^+), \dot{\mathbf{q}}(t^+)) \right) \hat{\mathbf{u}}(t^+) dt \\ & = - \left(\dot{\boldsymbol{\nu}}(t^-) \mathbf{M}(\mathbf{q}(t^+)) + \boldsymbol{\nu}(t^-) \dot{\mathbf{M}}(\mathbf{q}(t^+), \mathbf{u}(t^+)) \right) \hat{\mathbf{q}}(t^+) \Big|_{t_i^+}^{t_i^-} \\ & + \boldsymbol{\nu}(t^-) \mathbf{M}(\mathbf{q}(t^+)) \hat{\mathbf{u}}(t^+) \Big|_{t_i^+}^{t_i^-} + \int_{t_i^+}^{t_i^-} \frac{d^2}{dt^2} (\boldsymbol{\nu}(t^-) \mathbf{M}(\mathbf{q}(t^+))) \hat{\mathbf{q}}(t^+) dt, \end{aligned} \quad (4.95)$$

where the second-order time derivative of the integrand multiplier is given by:

$$\begin{aligned} & \frac{d^2}{dt^2} (\boldsymbol{\nu}(t^-) \mathbf{M}(\mathbf{q}(t^+))) \\ & = \ddot{\boldsymbol{\nu}}(t^-) \mathbf{M}(\mathbf{q}(t^+)) + 2 \dot{\boldsymbol{\nu}}(t^-) \dot{\mathbf{M}}(\mathbf{q}(t^+), \mathbf{u}(t^+)) + \boldsymbol{\nu}(t^-) \ddot{\mathbf{M}}(\mathbf{q}(t^+), \mathbf{u}(t^+), \dot{\mathbf{u}}(t^+)). \end{aligned}$$

Similarly, in order to eliminate the variations with respect to $\hat{\mathbf{u}}(t^+)$ on every open interval (t_i^+, t_{i+1}^-) such that $\{t_i, t_{i+1}\} \in \mathcal{I}_T$ the Lemma of Du Bois-Reymond is applied:

$$\begin{aligned} & \int_{t_i^+}^{t_i^-} \lambda(t^+) (\partial_{\mathbf{u}} g(\mathbf{q}(t^+), \mathbf{u}(t^+), \boldsymbol{\tau}(t^+)) - \boldsymbol{\nu}(t^-) \nabla_{\mathbf{u}} \mathbf{h}(\mathbf{q}(t^+), \mathbf{u}(t^+))) \hat{\mathbf{u}}(t^+) dt \\ &= (\lambda(t^+) \partial_{\mathbf{u}} g(\mathbf{q}(t^+), \mathbf{u}(t^+), \boldsymbol{\tau}(t^+)) - \boldsymbol{\nu}(t^-) \nabla_{\mathbf{u}} \mathbf{h}(\mathbf{q}(t^+), \mathbf{u}(t^+))) \hat{\mathbf{q}}(t^+) \Big|_{t_i^+}^{t_i^-} \\ &- \int_{t_i^+}^{t_i^-} \frac{d}{dt} (\lambda(t^+) \partial_{\mathbf{u}} g(\mathbf{q}(t^+), \mathbf{u}(t^+), \boldsymbol{\tau}(t^+)) - \boldsymbol{\nu}(t^-) \nabla_{\mathbf{u}} \mathbf{h}(\mathbf{q}(t^+), \mathbf{u}(t^+))) \hat{\mathbf{q}}(t^+) dt, \end{aligned} \quad (4.96)$$

where

$$\frac{d}{dt} (\partial_{\mathbf{u}} g(\mathbf{q}(t), \mathbf{u}(t), \boldsymbol{\tau}(t))) = \partial_{\mathbf{uu}}^2 g(\mathbf{q}(t), \mathbf{u}(t), \boldsymbol{\tau}(t)) \dot{\mathbf{u}}(t) + \partial_{\mathbf{uq}}^2 g(\mathbf{q}(t), \mathbf{u}(t), \boldsymbol{\tau}(t)) \mathbf{u}(t),$$

and

$$\frac{d}{dt} \nabla_{\mathbf{u}} \mathbf{h}(\mathbf{q}(t), \mathbf{u}(t)) = \nabla_{\mathbf{uu}}^2 \mathbf{h}(\mathbf{q}(t), \mathbf{u}(t)) \dot{\mathbf{u}}(t) + \nabla_{\mathbf{uq}}^2 \mathbf{h}(\mathbf{q}(t), \mathbf{u}(t)) \mathbf{u}(t).$$

The boundary terms obtained by the application of the Lemma of Du Bois-Reymond in equations (4.95) and (4.96) can be combined in shorthand notation as given in (4.97):

$$\boldsymbol{\nu}(t) \mathbf{M} \hat{\mathbf{u}}(t) + \left(\lambda(t) \partial_{\mathbf{u}} g - \boldsymbol{\nu}(t) \nabla_{\mathbf{u}} \mathbf{h} - \left(\dot{\boldsymbol{\nu}}(t) \mathbf{M} + \boldsymbol{\nu}(t) \dot{\mathbf{M}} \right) \right) \hat{\mathbf{q}}(t) \Big|_{t_i^+}^{t_i^-}. \quad (4.97)$$

The resulting boundary terms as given in (4.97) obtained by the application of the Lemma of Du Bois-Reymond are rearranged by making use of the equations (4.88) to (4.91) in the following form:

$$\begin{aligned} & \sum_{\forall \hat{\psi}_i \in \hat{\mathcal{V}}_i} \left(\int_{t_i^+}^{t_{i+1}^-} L_i(\mathbf{q}(s), \mathbf{u}(s), \dot{\mathbf{u}}(s)) ds + \int_{t_{i-1}^+}^{t_i^-} L_i(\mathbf{q}(s), \mathbf{u}(s), \dot{\mathbf{u}}(s)) ds \right)^\uparrow (\cdot; \hat{\psi}_i) = \\ & \boldsymbol{\Omega}_i^- \hat{\mathbf{q}}_i^- - \boldsymbol{\Omega}_i^+ \hat{\mathbf{q}}_i^+ + \boldsymbol{\Upsilon}_i^- \hat{t}_i^- - \boldsymbol{\Upsilon}_i^+ \hat{t}_i^+ - \boldsymbol{\nu}(t_i^-) \mathbf{M}(\mathbf{q}(t_i^-)) \hat{\boldsymbol{\chi}}_i^- - \boldsymbol{\nu}(t_i^+) \mathbf{M}(\mathbf{q}(t_i^+)) \hat{\mathbf{u}}_i^+ \\ & + \boldsymbol{\nu}(t_i^-) \mathbf{M}(\mathbf{q}(t_i^-)) \hat{\mathbf{u}}_i^- + \boldsymbol{\nu}(t_i^+) \mathbf{M}(\mathbf{q}(t_i^+)) \hat{\boldsymbol{\chi}}_i^+. \end{aligned} \quad (4.98)$$

Here $\boldsymbol{\Omega}_i$ and $\boldsymbol{\Upsilon}_i$ are defined by

$$\boldsymbol{\Omega}_i = \lambda(t_i) \partial_{\mathbf{u}(t_i)} g - (\dot{\boldsymbol{\nu}}(t_i) \mathbf{M} + \boldsymbol{\nu}(t_i) \dot{\mathbf{M}}) - \boldsymbol{\nu}(t_i) \nabla_{\mathbf{u}(t_i)} \mathbf{h}_i, \quad (4.99)$$

$$\begin{aligned} \boldsymbol{\Upsilon}_i &= -\boldsymbol{\nu}(t_i) \mathbf{M} \dot{\mathbf{u}}(t_i) \\ &- \left(\lambda(t_i) \partial_{\mathbf{u}(t_i)} g - \boldsymbol{\nu}(t_i) \nabla_{\mathbf{u}(t_i)} \mathbf{h} - \left(\dot{\boldsymbol{\nu}}(t_i) \mathbf{M} + \boldsymbol{\nu}(t_i) \dot{\mathbf{M}} \right) \right) \mathbf{u}(t_i). \end{aligned} \quad (4.100)$$

As can be seen in (4.98) the variations in $\hat{\boldsymbol{\chi}}_i^-$ and $\hat{\boldsymbol{\chi}}_i^+$ are linearly dependent on the variations $\hat{\mathbf{u}}_i^-$ and $\hat{\mathbf{u}}_i^+$, respectively. In passing from (4.97) to (4.98) proposition 2.1 is used where necessary. Since the time-derivatives and gradients of $\dot{\mathbf{M}}$, $\ddot{\mathbf{M}}$, $\nabla_{\mathbf{u}} \mathbf{h}$, g , $\boldsymbol{\Omega}_i$ and $\boldsymbol{\Upsilon}_i$ involve the generalised velocities and accelerations of the system at pre-, and post-transition state, the right superscripted

signs denote whether the pre-transition or post-transition values of the relevant entities are meant where these entities exist in the limit sense as stated in proposition 2.1. By the presence of transition sets that impose restrictions on the pre-, and post-transition generalised velocities and positions, the internal boundary variations are embedded into the discontinuous transversality conditions. By the tangential regularity of the transition sets, the sum of the indicator functions of the transition sets are rendered subdifferentially regular. By theorem (3) in [84] (Theorem 4.1.3) following is valid:

$$\left(\Psi_{\mathcal{C}_{\mathcal{I}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{I}_i}^+} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^+}\right)^\uparrow(\cdot; \hat{\psi}) = \Psi_{\mathcal{C}_{\mathcal{I}_i}^-}^\uparrow(\cdot; \hat{\psi}) + \Psi_{\mathcal{C}_{\mathcal{T}_i}^-}^\uparrow(\cdot; \hat{\psi}) + \Psi_{\mathcal{C}_{\mathcal{I}_i}^+}^\uparrow(\cdot; \hat{\psi}) + \Psi_{\mathcal{C}_{\mathcal{T}_i}^+}^\uparrow(\cdot; \hat{\psi}), \quad (4.101)$$

$\forall \hat{\psi} \in \hat{\mathcal{V}}$. The equality in (4.101) arises from the subdifferential regularity of the tangential regularity of the sets given in (4.54). Considering together with (4.98), the upper subderivative of the value functional J in the direction $\hat{\mathbf{u}}_i^+$ is given by:

$$J^\uparrow(\cdot; \hat{\mathbf{u}}_i^+) = \left(\Psi_{\mathcal{C}_{\mathcal{I}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{I}_i}^+} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^+}\right)^\uparrow(\cdot; \hat{\mathbf{u}}_i^+) - \boldsymbol{\nu}(\mathbf{t}_i^+) \mathbf{M} \hat{\mathbf{u}}_i^+. \quad (4.102)$$

The condition of optimality stated in (4.82) becomes:

$$J^\uparrow(\cdot; \hat{\mathbf{u}}_i^+) \geq 0 \Leftrightarrow \left(\Psi_{\mathcal{C}_{\mathcal{I}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{I}_i}^+} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^+}\right)^\uparrow(\cdot; \hat{\psi}) \geq \boldsymbol{\nu}(\mathbf{t}_i^+) \mathbf{M} \hat{\mathbf{u}}_i^+, \quad \forall \hat{\mathbf{u}}_i^+ \in \mathbb{R}^n. \quad (4.103)$$

Under the given regularity assumptions on the sets $\mathcal{C}_T$ and $\mathcal{C}_I$, this equivalently means that the vector $\boldsymbol{\nu}(\mathbf{t}_i^+) \mathbf{M}(\mathbf{q}(\mathbf{t}_i^+))$ is in the partial asymptotic limiting subdifferential of the indicator function of the set $(\mathcal{C}_{\mathcal{I}_i}^- \cup \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cup \mathcal{C}_{\mathcal{T}_i}^+)$ with respect to $\mathbf{u}(\mathbf{t}_i^+)$:

$$\boldsymbol{\nu}(\mathbf{t}_i^+) \mathbf{M}(\mathbf{q}(\mathbf{t}_i)) \in \partial_{\mathbf{u}(\mathbf{t}_i^+)}^\infty \Psi_{(\mathcal{C}_{\mathcal{I}_i}^- \cup \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cup \mathcal{C}_{\mathcal{T}_i}^+)}, \quad (4.104)$$

which equivalently can be expressed as:

$$\boldsymbol{\nu}(\mathbf{t}_i^+) \mathbf{M}(\mathbf{q}(\mathbf{t}_i)) \in \partial_{\mathbf{u}(\mathbf{t}_i^+)}^\infty \Psi_{(\mathcal{C}_{\mathcal{I}_i}^- \cap \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cap \mathcal{C}_{\mathcal{T}_i}^+)}, \quad (4.105)$$

by theorem (4) in [85]. By the properties of indicator functions the optimality condition states that the vector $\boldsymbol{\nu}(\mathbf{t}_i^+) \mathbf{M}(\mathbf{q}(\mathbf{t}_i^+))$ is in the partial singular limiting normal cone of the indicator function of the set $(\mathcal{C}_{\mathcal{I}_i}^- \cup \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cup \mathcal{C}_{\mathcal{T}_i}^+)$ with respect to $\hat{\mathbf{u}}(\mathbf{t}_i^+)$:

$$\boldsymbol{\nu}(\mathbf{t}_i^+) \mathbf{M}(\mathbf{q}(\mathbf{t}_i)) \in {}^\infty \mathcal{N}_{(\mathcal{C}_{\mathcal{I}_i}^- \cup \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cup \mathcal{C}_{\mathcal{T}_i}^+)}(\cdot; \hat{\mathbf{u}}(\mathbf{t}_i^+)). \quad (4.106)$$

Considering together with (4.98), the upper subderivative of the value functional J in the direction $\hat{\mathbf{u}}_i^-$ is given by:

$$J^\uparrow(\cdot; \hat{\mathbf{u}}_i^-) = \left(\Psi_{\mathcal{C}_{\mathcal{I}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{I}_i}^+} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^+}\right)^\uparrow(\cdot; \hat{\mathbf{u}}_i^-) + \boldsymbol{\nu}(\mathbf{t}_i^-) \mathbf{M} \hat{\mathbf{u}}_i^-. \quad (4.107)$$

The condition of optimality stated in (4.82) becomes:

$$J^\uparrow(\cdot; \hat{\mathbf{u}}_i^-) \geq 0 \Leftrightarrow \left(\Psi_{\mathcal{C}_{\mathcal{I}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{I}_i}^+} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^+} \right)^\uparrow(\cdot; \hat{\mathbf{u}}_i^-) \geq -\boldsymbol{\nu}(\mathbf{t}_i^-) \mathbf{M} \hat{\mathbf{u}}_i^-, \quad \forall \hat{\mathbf{u}}_i^- \in \mathbb{R}^n. \quad (4.108)$$

Under the given regularity assumptions on the sets $\mathcal{C}_T$ and $\mathcal{C}_{\mathcal{I}_i}$, this equivalently means that the vector $-\boldsymbol{\nu}(\mathbf{t}_i^-) \mathbf{M}(\mathbf{q}(\mathbf{t}_i^-))$ is in the partial asymptotic limiting subdifferential of the indicator function of the set $(\mathcal{C}_{\mathcal{I}_i}^- \cup \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cup \mathcal{C}_{\mathcal{T}_i}^+)$ with respect to $\mathbf{u}(\mathbf{t}_i^+)$:

$$-\boldsymbol{\nu}(\mathbf{t}_i^-) \mathbf{M}(\mathbf{q}(\mathbf{t}_i)) \in \partial_{\mathbf{u}(\mathbf{t}_i^-)}^\infty \Psi_{(\mathcal{C}_{\mathcal{I}_i}^- \cup \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cup \mathcal{C}_{\mathcal{T}_i}^+)}, \quad (4.109)$$

which equivalently can be expressed as:

$$\boldsymbol{\nu}(\mathbf{t}_i^-) \mathbf{M}(\mathbf{q}(\mathbf{t}_i)) \in \partial_{\mathbf{u}(\mathbf{t}_i^-)}^\infty \Psi_{(\mathcal{C}_{\mathcal{I}_i}^- \cap \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cap \mathcal{C}_{\mathcal{T}_i}^+)}, \quad (4.110)$$

by theorem (4) in [85]. By the properties of indicator functions the optimality condition states that the vector $-\boldsymbol{\nu}(\mathbf{t}_i^-) \mathbf{M}(\mathbf{q}(\mathbf{t}_i^-))$ is in the partial singular limiting normal cone of the indicator function of the set $(\mathcal{C}_{\mathcal{I}_i}^- \cup \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cup \mathcal{C}_{\mathcal{T}_i}^+)$ with respect to $\mathbf{u}(\mathbf{t}_i^-)$:

$$\boldsymbol{\nu}(\mathbf{t}_i^+) \mathbf{M}(\mathbf{q}(\mathbf{t}_i)) \in {}^\infty \mathcal{N}_{(\mathcal{C}_{\mathcal{I}_i}^- \cup \mathcal{C}_{\mathcal{I}_i}^+) \cap (\mathcal{C}_{\mathcal{T}_i}^- \cup \mathcal{C}_{\mathcal{T}_i}^+)}(\cdot; \mathbf{u}(\mathbf{t}_i^-)). \quad (4.111)$$

Considering together with (4.98), the upper subderivative of the value functional J in direction $\hat{\mathbf{q}}_i^+$ is given by:

$$\begin{aligned} J^\uparrow(\cdot; \hat{\mathbf{q}}_i^+) &= \left(\Psi_{\mathcal{C}_{\mathcal{I}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{I}_i}^+} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^+} \right)^\uparrow(\cdot; \hat{\mathbf{q}}_i^+) \\ &+ \left(-\lambda(\mathbf{t}_i^+) \partial_{\mathbf{q}(\mathbf{t}_i^+)} g(\mathbf{q}(\mathbf{t}_i^+), \mathbf{u}(\mathbf{t}_i^+), \boldsymbol{\tau}(\mathbf{t}_i^+)) + (\dot{\boldsymbol{\nu}}(\mathbf{t}_i^+) \mathbf{M} + \boldsymbol{\nu}(\mathbf{t}_i^+) \dot{\mathbf{M}}) + \boldsymbol{\nu}(\mathbf{t}_i^+) \nabla_{\mathbf{u}} \mathbf{h}_i \right) \hat{\mathbf{q}}_i^+. \end{aligned} \quad (4.112)$$

Considering together with (4.98), the upper subderivative of the value functional J in direction $\hat{\mathbf{q}}_i^-$ is given by:

$$\begin{aligned} J^\uparrow(\cdot; \hat{\mathbf{q}}_i^-) &= \left(\Psi_{\mathcal{C}_{\mathcal{I}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^-} + \Psi_{\mathcal{C}_{\mathcal{I}_i}^+} + \Psi_{\mathcal{C}_{\mathcal{T}_i}^+} \right)^\uparrow(\cdot; \hat{\mathbf{q}}_i^-) \\ &- \left(-\lambda(\mathbf{t}_i^-) \partial_{\mathbf{q}(\mathbf{t}_i^-)} g(\mathbf{q}(\mathbf{t}_i^-), \mathbf{u}(\mathbf{t}_i^-), \boldsymbol{\tau}(\mathbf{t}_i^-)) + (\dot{\boldsymbol{\nu}}(\mathbf{t}_i^-) \mathbf{M} + \boldsymbol{\nu}(\mathbf{t}_i^-) \dot{\mathbf{M}}) + \boldsymbol{\nu}(\mathbf{t}_i^-) \nabla_{\mathbf{u}} \mathbf{h}_i \right) \hat{\mathbf{q}}_i^-. \end{aligned} \quad (4.113)$$

As a corollary of assumption 4.3.2 the post and pre-transition variations of the generalised position are set equal:

$$\hat{\mathbf{q}}_i^+ = \hat{\mathbf{q}}_i^- = \hat{\mathbf{q}}_i. \quad (4.114)$$

The combination of the conditions given in (4.112) and (4.113) under assumption 4.3.2 reveals following optimality condition in variational inequality form:

$$\begin{aligned} J^\uparrow(\cdot; \hat{\mathbf{q}}_i) &= J^\uparrow(\cdot; \hat{\mathbf{q}}_i^-) + J^\uparrow(\cdot; \hat{\mathbf{q}}_i^+) \geq 0, \quad \forall (\hat{\mathbf{q}}_i, \hat{\mathbf{q}}_i^-, \hat{\mathbf{q}}_i^+) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \\ &\text{such that } \hat{\mathbf{q}}_i^+ = \hat{\mathbf{q}}_i^- = \hat{\mathbf{q}}_i. \end{aligned} \quad (4.115)$$

which is equivalent to:

$$\left(\Psi_{\mathcal{C}_{\mathbf{i}_i}^-} + \Psi_{\mathcal{C}_{\mathbf{T}_i}^-} + \Psi_{\mathcal{C}_{\mathbf{i}_i}^+} + \Psi_{\mathcal{C}_{\mathbf{T}_i}^+} \right)^\uparrow (\cdot; \hat{\mathbf{q}}_i) \geq (\boldsymbol{\Omega}_i^+ - \boldsymbol{\Omega}_i^-) \hat{\mathbf{q}}_i, \quad \forall \hat{\mathbf{q}}_i \in \mathbb{R}^n. \quad (4.116)$$

Under the given regularity assumptions on the sets $\mathcal{C}_T$ and $\mathcal{C}_i$, this equivalently means that the vector $\boldsymbol{\Omega}_i^+ - \boldsymbol{\Omega}_i^-$ is in the partial asymptotic limiting subdifferential of the set $(\mathcal{C}_{\mathbf{i}_i}^- \cup \mathcal{C}_{\mathbf{i}_i}^+) \cap (\mathcal{C}_{\mathbf{T}_i}^- \cup \mathcal{C}_{\mathbf{T}_i}^+)$ with respect to $\mathbf{q}(t_i)$ at the optimal solution:

$$\boldsymbol{\Omega}_i^+ - \boldsymbol{\Omega}_i^- \in \partial_{\mathbf{q}(t_i)}^\infty \Psi_{(\mathcal{C}_{\mathbf{i}_i}^- \cup \mathcal{C}_{\mathbf{i}_i}^+) \cap (\mathcal{C}_{\mathbf{T}_i}^- \cup \mathcal{C}_{\mathbf{T}_i}^+)}, \quad (4.117)$$

which equivalently can be expressed as:

$$\boldsymbol{\Omega}_i^+ - \boldsymbol{\Omega}_i^- \in \partial_{\mathbf{q}(t_i)}^\infty \Psi_{(\mathcal{C}_{\mathbf{i}_i}^- \cap \mathcal{C}_{\mathbf{i}_i}^+) \cap (\mathcal{C}_{\mathbf{T}_i}^- \cap \mathcal{C}_{\mathbf{T}_i}^+)}. \quad (4.118)$$

By the properties of indicator functions the optimality condition states that the vector $\boldsymbol{\Omega}_i^+ - \boldsymbol{\Omega}_i^-$ is in the partial singular limiting normal cone of the indicator function of the set $(\mathcal{C}_{\mathbf{i}_i}^- \cup \mathcal{C}_{\mathbf{i}_i}^+) \cap (\mathcal{C}_{\mathbf{T}_i}^- \cup \mathcal{C}_{\mathbf{T}_i}^+)$ with respect to $\mathbf{q}(t_i)$:

$$\boldsymbol{\Omega}_i^+ - \boldsymbol{\Omega}_i^- \in {}^\infty \mathcal{N}_{(\mathcal{C}_{\mathbf{i}_i}^- \cup \mathcal{C}_{\mathbf{i}_i}^+) \cap (\mathcal{C}_{\mathbf{T}_i}^- \cup \mathcal{C}_{\mathbf{T}_i}^+)}(\cdot; \mathbf{q}(t_i)). \quad (4.119)$$

The upper subderivatives of the value function with respect to impulsive controls $\hat{\boldsymbol{\zeta}}_i^+$ and $\hat{\boldsymbol{\zeta}}_i^-$ yields following variational inequalities as optimality conditions:

$$J^\uparrow(\cdot; \hat{\boldsymbol{\zeta}}_i^+) = \left(\Psi_{\mathcal{C}_{\mathbf{i}_i}^-} + \Psi_{\mathcal{C}_{\mathbf{T}_i}^-} + \Psi_{\mathcal{C}_{\mathbf{i}_i}^+} + \Psi_{\mathcal{C}_{\mathbf{T}_i}^+} \right)^\uparrow (\cdot; \hat{\boldsymbol{\zeta}}_i^+) \geq 0, \quad \forall \hat{\boldsymbol{\zeta}}_i^+ \in \mathbb{R}^n, \quad (4.120)$$

and

$$J^\uparrow(\cdot; \hat{\boldsymbol{\zeta}}_i^-) = \left(\Psi_{\mathcal{C}_{\mathbf{i}_i}^-} + \Psi_{\mathcal{C}_{\mathbf{T}_i}^-} + \Psi_{\mathcal{C}_{\mathbf{i}_i}^+} + \Psi_{\mathcal{C}_{\mathbf{T}_i}^+} \right)^\uparrow (\cdot; \hat{\boldsymbol{\zeta}}_i^-) \geq 0, \quad \forall \hat{\boldsymbol{\zeta}}_i^- \in \mathbb{R}^n, \quad (4.121)$$

respectively. The optimality conditions given (4.120) and (4.121) translate by theorem by theorem 4 in [85] into following normal cone inclusions:

$$(\boldsymbol{\xi}_i^+ + \boldsymbol{\xi}_i^-) \mathbf{B}_i(\mathbf{q}(t_i)) \in {}^\infty \mathcal{N}_{(\mathcal{C}_{\mathbf{i}_i}^- \cup \mathcal{C}_{\mathbf{i}_i}^+) \cap (\mathcal{C}_{\mathbf{T}_i}^- \cup \mathcal{C}_{\mathbf{T}_i}^+)}(\cdot; \boldsymbol{\zeta}_i^-), \quad (4.122)$$

and

$$-(\boldsymbol{\xi}_i^+ + \boldsymbol{\xi}_i^-) \mathbf{B}_i(\mathbf{q}(t_i)) \in {}^\infty \mathcal{N}_{(\mathcal{C}_{\mathbf{i}_i}^- \cup \mathcal{C}_{\mathbf{i}_i}^+) \cap (\mathcal{C}_{\mathbf{T}_i}^- \cup \mathcal{C}_{\mathbf{T}_i}^+)}(\cdot; \boldsymbol{\zeta}_i^+), \quad (4.123)$$

for some $\boldsymbol{\xi}_i^+ \in \mathbb{R}^{1 \times n}$ and $\boldsymbol{\xi}_i^- \in \mathbb{R}^{1 \times n}$.

By corollary 5 of theorem 3 in [84] the partial normal cones given in (4.106), (4.111), (4.119), (4.122) and (4.123) which are obtained from the variational inequalities (4.103), (4.108), (4.116), (4.120) and (4.121) via theorem 4 in [85] can be stated due to the tangential regularity of the sets given (4.54) as a normal cone inclusion of the form:

$$\begin{bmatrix} \nu(t_i^+) \mathbf{M}(\mathbf{q}(t_i)) \\ -\nu(t_i^-) \mathbf{M}(\mathbf{q}(t_i)) \\ \Omega_i^+ - \Omega_i^- \\ (\xi_i^+ + \xi_i^-) \mathbf{B}(\mathbf{q}(t_i)) \\ -(\xi_i^+ + \xi_i^-) \mathbf{B}(\mathbf{q}(t_i)) \end{bmatrix} \in {}^\infty \mathcal{N}_{(c_{I_i}^- \cap c_{I_i}^+) \cap (c_{T_i}^- \cap c_{T_i}^+)}(\cdot; \mathbf{y}(t_i)) \quad (4.124)$$

and the vector $\mathbf{y}(t_i)$ is given by:

$$\mathbf{y}(t_i) = \begin{bmatrix} \mathbf{u}(t_i^+), & \mathbf{u}(t_i^-), & \mathbf{q}(t_i), & \zeta_i^+, & \zeta_i^- \end{bmatrix}^T. \quad (4.125)$$

The variations due to the transition time are some more involved. The subderivatives with respect to pre-transitional time and post-transitional time instant yield (4.126) and (4.127) as variational inequalities as optimality condition:

$$J^\uparrow(\cdot; \hat{t}_i^-) = \left(\Psi_{c_{I_i}^-} + \Psi_{c_{T_i}^-} + \Psi_{c_{I_i}^+} + \Psi_{c_{T_i}^+} \right)^\uparrow(\cdot; \hat{t}_i^-) + \Upsilon^- \hat{t}_i^- \geq 0, \quad \forall \hat{t}_i^- \in \mathbb{R}, \quad (4.126)$$

and

$$J^\uparrow(\cdot; \hat{t}_i^+) = \left(\Psi_{c_{I_i}^-} + \Psi_{c_{T_i}^-} + \Psi_{c_{I_i}^+} + \Psi_{c_{T_i}^+} \right)^\uparrow(\cdot; \hat{t}_i^+) + \Upsilon^+ \hat{t}_i^+ \geq 0, \quad \forall \hat{t}_i^+ \in \mathbb{R}. \quad (4.127)$$

The entities Υ^+ and Υ^- are defined as given in (4.98). As a corollary of assumption 4.3.3 the post and pre-transition variations of the transition instant are set equal:

$$\hat{t}_i^+ = \hat{t}_i^- = \hat{t}_i. \quad (4.128)$$

The condition

$$J^\uparrow(\cdot; \hat{t}_i) = J^\uparrow(\cdot; \hat{t}_i^-) + J^\uparrow(\cdot; \hat{t}_i^+) \geq 0 \quad \forall (\hat{t}_i, \hat{t}_i^-, \hat{t}_i^+) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ \text{such that } \hat{t}_i^+ = \hat{t}_i^- = \hat{t}_i. \quad (4.129)$$

reveals following variational inequality under consideration of affine relations from (4.88) to (4.91) and theorem 3 in [84]:

$$- \sum_{\forall \theta \in \Theta} \left\langle \left(\Psi_{c_{I_i}^-} + \Psi_{c_{T_i}^-} + \Psi_{c_{I_i}^+} + \Psi_{c_{T_i}^+} \right)^\uparrow(\cdot; \hat{\theta}), \hat{\theta} \right\rangle \hat{t}_i + (\Upsilon^+ + \Upsilon^-) \hat{t}_i \geq 0, \quad \forall \hat{t}_i \in \mathbb{R}. \quad (4.130)$$

where the index set Θ is given by:

$$\Theta = \{ \mathbf{u}(t_i^+), \mathbf{u}(t_i^-), \mathbf{q}(t_i^-), \mathbf{q}(t_i^+) \}. \quad (4.131)$$

4.5.3 Total Directional Derivatives of The Value Function J

Total Directional derivative in direction $\hat{\mathbf{q}}(t^+)$ Combination of equations (4.55), (4.95) and (4.96) reveal the total directional derivative in direction $\hat{\mathbf{q}}(t^+)$ and the optimality condition can be expressed as the following variational inequality:

$$J^\uparrow(\cdot, \hat{\mathbf{q}}(t^+)) = \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \Psi \hat{\mathbf{q}}(t^+) dt \geq 0, \quad \forall t \in \text{int dom}(P_i). \quad (4.132)$$

where Ψ is given by:

$$\Psi = \nu(t^-) \left(\nabla_{\mathbf{q}} S_i^+ + \ddot{\mathbf{M}} + \frac{d}{dt} \nabla_{\mathbf{u}} \mathbf{h}_i \right) + \ddot{\nu}(t^-) \mathbf{M} + 2\dot{\nu}(t^-) \dot{\mathbf{M}} + \dot{\nu}(t^-) \nabla_{\mathbf{u}} \mathbf{h}_i \quad (4.133)$$

Total Directional derivative in direction $\hat{\boldsymbol{\tau}}(t^+)$ The total directional derivative in direction $\hat{\boldsymbol{\tau}}(t^+)$ and the optimality condition can be expressed as the following variational inequality:

$$J^\uparrow(\cdot, \hat{\boldsymbol{\tau}}(t^+)) = \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \nu(t^-) \nabla_{\boldsymbol{\tau}} S_i^+ \hat{\boldsymbol{\tau}}(t^+) + \Psi_{\mathcal{C}_{\boldsymbol{\tau}}}^\uparrow(\cdot, \hat{\boldsymbol{\tau}}(t^+)) dt, \quad \forall t \in \text{int dom}(P_i). \quad (4.134)$$

Total Directional derivative in direction $\hat{\mathbf{u}}_f$ The total directional derivative in direction $\hat{\mathbf{u}}_f$ and the optimality condition can be expressed as the following variational inequality:

$$J^\uparrow(\cdot, \hat{\mathbf{u}}_f) = \nu(t_f) \mathbf{M} \hat{\mathbf{u}}_f + \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{\mathbf{u}}_f) \geq 0. \quad (4.135)$$

Total Directional derivative in direction $\hat{\mathbf{q}}_f$ The total directional derivative in direction $\hat{\mathbf{q}}_f$ and the optimality condition can be expressed as the following variational inequality:

$$J^\uparrow(\cdot, \hat{\mathbf{q}}_f) = \left[-(\dot{\nu}(t_f) \mathbf{M} + \nu(t_f) \dot{\mathbf{M}} - \nu(t_f) \nabla_{\mathbf{u}} \mathbf{h}_i) \right] \hat{\mathbf{q}}_f + \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{\mathbf{q}}_f) \geq 0. \quad (4.136)$$

Total Directional derivative in direction $\hat{\mathbf{t}}_f$ The total directional derivative in direction $\hat{\mathbf{t}}_f$ and the optimality condition can be expressed as the following variational inequality:

$$\begin{aligned} (\nu(t_f) \nabla_{\mathbf{u}} \mathbf{h}_i \mathbf{u}(t_f) - \nu(t_f) \mathbf{M} \dot{\mathbf{u}}(t_f) + (\dot{\nu}(t_f) \mathbf{M} + \nu(t_f) \dot{\mathbf{M}}) \mathbf{u}(t_f) t_f \\ + \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \mathbf{u}(t_f) \hat{\mathbf{t}}_f) + \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \dot{\mathbf{u}}(t_f) \hat{\mathbf{t}}_f) \geq 0. \end{aligned} \quad (4.137)$$

The smooth structure of the sets $\mathcal{C}_i^+$ and $\mathcal{C}_i^-$, which are $\mathcal{C}^1$ in their arguments, enable further evaluation of directional derivatives, in order to evaluate the variational inequalities. The directional derivatives of $\Psi_{\mathcal{C}_i^+}$ with respect to $\hat{\mathbf{q}}(t_i^-)$, $\hat{\mathbf{u}}(t_i^+)$, $\hat{\mathbf{u}}(t_i^-)$, ζ_i^+ , ζ_i^- are given as:

$$\Psi_{\mathcal{C}_i^+}^\uparrow (\cdot, \hat{\mathbf{q}}(t_i^-)) = 0, \quad \forall t_i \in \mathcal{I}_T, \quad (4.138)$$

$$\Psi_{\mathcal{C}_i^+}^\uparrow (\cdot, \hat{\mathbf{u}}(t_i^+)) = \boldsymbol{\xi}_i^+ \mathbf{M}(\mathbf{q}(t_i^+)) \hat{\mathbf{u}}(t_i^+), \quad \forall t_i \in \mathcal{I}_T, \quad (4.139)$$

$$\Psi_{\mathcal{C}_i^+}^\uparrow (\cdot, \hat{\mathbf{u}}(t_i^-)) = -\boldsymbol{\xi}_i^+ \mathbf{M}(\mathbf{q}(t_i^+)) \hat{\mathbf{u}}(t_i^-), \quad \forall t_i \in \mathcal{I}_T, \quad (4.140)$$

$$\Psi_{\mathcal{C}_i^+}^\uparrow (\cdot, \zeta_i^+) = \boldsymbol{\xi}_i^+ \sum_{j \in \mathcal{I}_f} \mathbf{b}_j(\mathbf{q}(t_i^+)) \zeta_i^+, \quad \forall t_i \in \mathcal{I}_T, \quad (4.141)$$

$$\Psi_{\mathcal{C}_i^+}^\uparrow (\cdot, \zeta_i^-) = -\boldsymbol{\xi}_i^+ \sum_{j \in \mathcal{I}_f} \mathbf{b}_j(\mathbf{q}(t_i^+)) \zeta_i^-, \quad \forall t_i \in \mathcal{I}_T. \quad (4.142)$$

The directional derivatives of $\Psi_{\mathcal{C}_i^-}$ with respect to $\hat{\mathbf{q}}(t_i^+)$, $\hat{\mathbf{u}}(t_i^+)$, $\hat{\mathbf{u}}(t_i^-)$, ζ_i^+ , ζ_i^- are given as:

$$\Psi_{\mathcal{C}_i^-}^\uparrow (\cdot, \hat{\mathbf{q}}(t_i^+)) = 0, \quad \forall t_i \in \mathcal{I}_T, \quad (4.143)$$

$$\Psi_{\mathcal{C}_i^-}^\uparrow (\cdot, \hat{\mathbf{u}}(t_i^+)) = \boldsymbol{\xi}_i^- \mathbf{M}(\mathbf{q}(t_i^-)) \hat{\mathbf{u}}(t_i^+), \quad \forall t_i \in \mathcal{I}_T, \quad (4.144)$$

$$\Psi_{\mathcal{C}_i^-}^\uparrow (\cdot, \hat{\mathbf{u}}(t_i^-)) = -\boldsymbol{\xi}_i^- \mathbf{M}(\mathbf{q}(t_i^-)) \hat{\mathbf{u}}(t_i^-), \quad \forall t_i \in \mathcal{I}_T, \quad (4.145)$$

$$\Psi_{\mathcal{C}_i^-}^\uparrow (\cdot, \zeta_i^+) = \boldsymbol{\xi}_i^- \mathbf{B}_i(\mathbf{q}(t_i^-)) \zeta_i^+, \quad \forall t_i \in \mathcal{I}_T, \quad (4.146)$$

$$\Psi_{\mathcal{C}_i^-}^\uparrow (\cdot, \zeta_i^-) = -\boldsymbol{\xi}_i^- \mathbf{B}_i(\mathbf{q}(t_i^-)) \zeta_i^-, \quad \forall t_i \in \mathcal{I}_T. \quad (4.147)$$

The directional derivatives of $\Psi_{\mathcal{C}_i^-}$ with respect to $\hat{\mathbf{q}}(t_i^-)$ and $\Psi_{\mathcal{C}_i^+}$ with respect to $\hat{\mathbf{q}}(t_i^+)$ are given as:

$$\Psi_{\mathcal{C}_i^+}^\uparrow (\cdot, \hat{\mathbf{q}}(t_i^+)) = \quad (4.148)$$

$$\boldsymbol{\xi}_i^+ \nabla_{\mathbf{q}} \left[\mathbf{M}(\mathbf{q}(t_i^+)) (\mathbf{u}(t_i^+) - \mathbf{u}(t_i^-)) - \sum_{j \in \mathcal{I}_f} \mathbf{b}_j(\mathbf{q}(t_i^+)) (\zeta_i^+ - \zeta_i^-) \right] \hat{\mathbf{q}}(t_i^+), \quad \forall t_i \in \mathcal{I}_T,$$

$$\Psi_{\mathcal{C}_i^-}^\uparrow (\cdot, \hat{\mathbf{q}}(t_i^-)) = \quad (4.149)$$

$$\boldsymbol{\xi}_i^- \nabla_{\mathbf{q}} \left[\mathbf{M}(\mathbf{q}(t_i^-)) (\mathbf{u}(t_i^+) - \mathbf{u}(t_i^-)) - \sum_{j \in \mathcal{I}_f} \mathbf{b}_j(\mathbf{q}(t_i^-)) (\zeta_i^+ - \zeta_i^-) \right] \hat{\mathbf{q}}(t_i^-), \quad \forall t_i \in \mathcal{I}_T.$$

Here $\boldsymbol{\xi}_i^+$ and $\boldsymbol{\xi}_i^-$ are dual multipliers.

4.6 Necessary Conditions

Under Assumptions 4.1, 4.2 and 4.3 the value function possesses regularity properties which enable the statement of "sharp" necessary conditions.

Theorem 4.6.1 [Yunt] Let assumptions 4.1, 4.2 and 4.3 be valid for the optimal control problem. If optimal trajectories of generalised positions $\mathbf{q}^*(t^+) \in \mathcal{AC}[\mathbb{R}^n]$, velocities $\mathbf{u}^*(t^+) \in$

$\mathcal{RCLBV}[\mathbb{R}^n]$ provide a minimum for the described optimal control problem, then there exist optimal controls $\boldsymbol{\tau}^*(t)$, optimal transition times $t_i^* \in \mathcal{I}_T$, dual multipliers $\boldsymbol{\xi}_i^{+*}, \boldsymbol{\xi}_i^{-*}, \boldsymbol{\alpha}_i^{+*}, \boldsymbol{\alpha}_i^{-*}$ in $\mathbb{R}^{1 \times n}$, $\forall t_i^* \in \mathcal{I}_T$, transition location triplets $\{\mathbf{q}^*(t_i), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)\}$, dual state $\boldsymbol{\nu}^*(t^-) \in \mathcal{LCLBV}^*[\mathbb{R}^{1 \times n}]$ (where $*$ denote dual space) and a scalar $\lambda(t^+) \in \{0, 1\}$, such that $\lambda^*(t^+) + |\boldsymbol{\nu}^*(t^-)| > 0$ for all $t \in \Omega_t \cup \mathcal{I}_T$, which fulfill:

1. the Lebesgue-measurable dynamics in every interval of motion $t \in (t_i^{+*}, t_{i+1}^{-*})$

$$\mathbf{M}(\mathbf{q}^*(t^+))\dot{\mathbf{u}}^*(t^+) - \mathbf{h}_i(\mathbf{q}^*(t^+), \mathbf{u}^*(t^+)) - \mathbf{B}_i(\mathbf{q}^*(t^+))\boldsymbol{\tau}^{+*} = \mathbf{0}, \quad \text{a.e.}, \quad (4.150)$$

2. the Lebesgue-measurable dual dynamics

$$\ddot{\boldsymbol{\nu}}^*(t^-)\mathbf{D}_i + \dot{\boldsymbol{\nu}}^*(t^-)\mathbf{E}_i + \boldsymbol{\nu}^*(t^-)\mathbf{F}_i + \mathbf{G}_i = \mathbf{0}, \quad \text{a.e.}, \quad t \in (t_i^{+*}, t_{i+1}^{-*}), \quad (4.151)$$

where the coefficients in the differential equation above are given by:

$$\begin{aligned} \mathbf{D}_i &= \mathbf{M}(\mathbf{q}^*(t^+)), \\ \mathbf{E}_i &= 2\dot{\mathbf{M}}(\mathbf{q}^*(t^+)) + \nabla_{\mathbf{u}} \mathbf{h}_i(\mathbf{q}^*(t^+), \mathbf{u}^*(t^+)), \\ \mathbf{F}_i &= \nabla_{\mathbf{q}}[\mathbf{M}(\mathbf{q}^*(t^+))\dot{\mathbf{u}}^*(t^+) - \mathbf{h}_i(\mathbf{q}^*(t^+), \mathbf{u}^*(t^+)) - \mathbf{B}_i(\mathbf{q}^*(t^+))\boldsymbol{\tau}^*] + \ddot{\mathbf{M}}(\mathbf{q}^*(t^+)) \\ &\quad + \frac{d}{dt}(\nabla_{\mathbf{u}}[\mathbf{h}_i(\mathbf{q}^*(t^+), \mathbf{u}^*(t^+))]), \\ \mathbf{G}_i &= \lambda^*(t^+) \left(\partial_{\mathbf{q}} g(\mathbf{q}^*(t^+), \mathbf{u}^*(t^+), \boldsymbol{\tau}^*(t^+)) - \frac{d}{dt} \partial_{\mathbf{u}} g(\mathbf{q}^*(t^+), \mathbf{u}^*(t^+), \boldsymbol{\tau}^*(t^+)) \right), \end{aligned}$$

3. the optimal control law on every interval (t_i^{+*}, t_{i+1}^{-*})

$$-\lambda^*(t^+)\partial_{\boldsymbol{\tau}} g(\mathbf{q}^*(t^+), \mathbf{u}^*(t^+), \boldsymbol{\tau}^*(t^+)) - \boldsymbol{\nu}^*(t^-)\mathbf{B}_i(\mathbf{q}^*(t^+)) \in \mathcal{N}_{\mathcal{C}_{\boldsymbol{\tau}}}(\boldsymbol{\tau}^{+*}), \quad \text{a.e.}, \quad (4.152)$$

4. the condition

$$\Upsilon_i^{+*} + \Upsilon_i^{-*} = \mathbf{r}_{i1} \mathbf{u}^*(t_i^-) + \mathbf{r}_{i2} \mathbf{u}^*(t_i^+) + \mathbf{r}_{i3} \dot{\mathbf{u}}^*(t_i^-) + \mathbf{r}_{i4} \dot{\mathbf{u}}^*(t_i^+), \quad \forall t_i^* \in \mathcal{I}_T, \quad (4.153)$$

where the vectors $\mathbf{r}_{i1}$, $\mathbf{r}_{i2}$, $\mathbf{r}_{i3}$ and $\mathbf{r}_{i4}$ are given by:

$$\begin{aligned} \mathbf{r}_{i1} &= \boldsymbol{\alpha}_i^{-*} \nabla_{\mathbf{q}(t_i)} \mathbf{p}_i^-(\mathbf{q}^*(t_i^-), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) \\ &\quad + \boldsymbol{\xi}_i^{-*} \nabla_{\mathbf{q}(t_i)} [\mathbf{M}(\mathbf{q}^*(t_i))(\mathbf{u}(t_i^{+*}) - \mathbf{u}(t_i^{-*})) - \mathbf{B}_i(\mathbf{q}^*(t_i))(\boldsymbol{\zeta}_i^{+*} - \boldsymbol{\zeta}_i^{-*})], \\ \mathbf{r}_{i2} &= \boldsymbol{\alpha}_i^{+*} \nabla_{\mathbf{q}(t_i)} \mathbf{p}_i^+(\mathbf{q}^*(t_i^+), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) \\ &\quad + \boldsymbol{\xi}_i^{+*} \nabla_{\mathbf{q}(t_i)} [\mathbf{M}(\mathbf{q}^*(t_i))(\mathbf{u}(t_i^{+*}) - \mathbf{u}(t_i^{-*})) - \mathbf{B}_i(\mathbf{q}^*(t_i))(\boldsymbol{\zeta}_i^{+*} - \boldsymbol{\zeta}_i^{-*})], \\ \mathbf{r}_{i3} &= \boldsymbol{\alpha}_i^{-*} \nabla_{\mathbf{u}(t_i^-)} \mathbf{p}_i^-(\mathbf{q}^*(t_i^-), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) \\ &\quad - \boldsymbol{\alpha}_i^{+*} \nabla_{\mathbf{u}(t_i^-)} \mathbf{p}_i^-(\mathbf{q}^*(t_i^-), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) - \boldsymbol{\xi}_i^{+*} \mathbf{M}(\mathbf{q}^*(t_i)) - \boldsymbol{\xi}_i^{-*} \mathbf{M}(\mathbf{q}^*(t_i)), \\ \mathbf{r}_{i4} &= \boldsymbol{\alpha}_i^{-*} \nabla_{\mathbf{u}(t_i^+)} \mathbf{p}_i^+(\mathbf{q}^*(t_i^+), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) \\ &\quad - \boldsymbol{\alpha}_i^{+*} \nabla_{\mathbf{u}(t_i^+)} \mathbf{p}_i^+(\mathbf{q}^*(t_i^+), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) + \boldsymbol{\xi}_i^{+*} \mathbf{M}(\mathbf{q}^*(t_i)) + \boldsymbol{\xi}_i^{-*} \mathbf{M}(\mathbf{q}^*(t_i)), \end{aligned}$$

5. the impact equation and transition conditions at a transition

$$\mathcal{C}_{T_i}^* = \mathcal{C}_{T_i}^{*+} \cup \mathcal{C}_{T_i}^{*-}, \quad \mathcal{C}_{I_i}^* = \mathcal{C}_{I_i}^{*+} \cup \mathcal{C}_{I_i}^{*-}, \quad \forall t_i^* \in \mathcal{I}_T, \quad (4.154)$$

6. the discontinuity conditions of the dual state $\boldsymbol{\nu}$ and of its time-derivative $\dot{\boldsymbol{\nu}}$ for all $t_i^* \in \mathcal{I}_T$:

$$\begin{aligned} & (\boldsymbol{\nu}^*(t_i^+) - \boldsymbol{\nu}^*(t_i^-)) \mathbf{M}(\mathbf{q}^*(t)) = \\ & \boldsymbol{\alpha}_i^{+*} \nabla_{\mathbf{u}(t_i^+)} (\mathbf{p}_i^+(\mathbf{q}^*(t_i^+), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) + \mathbf{p}_i^-(\mathbf{q}^*(t_i^-), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-))) \\ & + \boldsymbol{\alpha}_i^{-*} \nabla_{\mathbf{u}(t_i^-)} (\mathbf{p}_i^+(\mathbf{q}^*(t_i^+), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) + \mathbf{p}_i^-(\mathbf{q}^*(t_i^-), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-))) , \end{aligned} \quad (4.155)$$

and

$$\begin{aligned} & \boldsymbol{\Omega}_i^{+*} - \boldsymbol{\Omega}_i^{-*} = \\ & - \nabla_{\mathbf{q}} [\boldsymbol{\alpha}_i^{+*} \mathbf{p}_i^+(\mathbf{q}^*(t_i^+), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)) - \boldsymbol{\alpha}_i^{-*} \mathbf{p}_i^-(\mathbf{q}^*(t_i^-), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-))] \\ & - (\boldsymbol{\xi}_i^{+*} + \boldsymbol{\xi}_i^{-*}) \nabla_{\mathbf{q}} [\mathbf{M}(\mathbf{u}^*(t_i^+) - \mathbf{u}^*(t_i^-)) - \mathbf{B}_i(\boldsymbol{\zeta}_i^{+*} - \boldsymbol{\zeta}_i^{-*})] , \end{aligned} \quad (4.156)$$

7. the impulsive optimal control law condition

$$(\boldsymbol{\xi}_i^{-*} + \boldsymbol{\xi}_i^{+*})^T \in \text{Ker} (\mathbf{B}_i(\mathbf{q}^*(t_i))^T), \quad \forall t_i^* \in \mathcal{I}_T, \quad (4.157)$$

8. the boundary constraints $\mathcal{C}_f$ at final state,

9. the variational inequality with respect to the variations at final time $\hat{t}_f$:

$$\left(\Upsilon_f^* - \langle \Psi_{\mathcal{C}_f}^\uparrow(\cdot; \hat{\mathbf{q}}(t_f)), \mathbf{u}(t_f) \rangle - \langle \Psi_{\mathcal{C}_f}^\uparrow(\cdot; \hat{\mathbf{u}}(t_f)), \dot{\mathbf{u}}(t_f) \rangle \right) \hat{t}_f \geq 0, \quad \forall \hat{t}_f \in \mathbb{R}, \quad (4.158)$$

10. the transversality condition at final state:

$$\begin{bmatrix} -\boldsymbol{\Omega}_f^* \\ -\boldsymbol{\nu}^*(t_f) \mathbf{M}(\mathbf{q}^*(t_f)) \end{bmatrix} \in {}^\infty \mathcal{N}_{\mathcal{C}_f}(\mathbf{q}^*(t_f), \mathbf{u}^*(t_f)) . \quad (4.159)$$

Since the derivatives and gradients of $\dot{\mathbf{M}}$, $\ddot{\mathbf{M}}$, $\nabla_{\mathbf{u}} \mathbf{h}$ involve the generalised velocities and accelerations of the system at pre-, and post-transition state, the right superscripted signs denote whether the pre-transition or post-transition values of the relevant entities are meant. The gradients and time derivatives of several tensors in the Einstein notation convention are given as follows:

$$\begin{aligned} \dot{m}_{ij} &= \nabla_{\mathbf{q}_k} m_{ij} \mathbf{u}_k(t), \\ \ddot{m}_{ij} &= \nabla_{\mathbf{q}_k \mathbf{q}_l}^2 m_{ij} \mathbf{u}_k(t) \mathbf{u}_l(t) + \nabla_{\mathbf{q}_k} m_{ij} \dot{\mathbf{u}}_k(t), \\ \frac{d}{dt} [\nabla_{\mathbf{u}_k} \mathbf{h}_p] &= \nabla_{\mathbf{u}_k \mathbf{u}_l}^2 \mathbf{h}_p \dot{\mathbf{u}}_l(t) + \nabla_{\mathbf{u}_k \mathbf{q}_l}^2 \mathbf{h}_p \mathbf{u}_l(t), \end{aligned}$$

where a_{ij} denotes the relevant element of a second-order tensor $\mathbf{A}$.

4.7 Necessary Transition Conditions for Mechanical Systems with Blockable DOF

In this section the necessary conditions for the time-optimal control problem of mechanical systems with blockable DOF is presented based on the theorem given in [113]. The relation between time-optimal control and impulsive control is emphasised, because in the philosophy of time-optimal control, it is taken advantage of any excessive control action in order to attain the goal and impulsive control action is the utmost excessive control action that can be applied to a dynamical system, since impulsive control forces can grow to infinity on a single time instant. The modeling of underactuated mechanical systems is based on section 2.6.

Theorem 4.7.1 [Yunt] Let assumptions 4.1, 4.2 and 4.3 be valid for the optimal control problem. If optimal trajectories of generalised positions $\mathbf{q}^*(t^+) \in \mathcal{AC}[\mathbb{R}^n]$, velocities $\mathbf{u}^*(t^+) \in \mathcal{RCLBV}[\mathbb{R}^n]$ provide a minimum for the described optimal control problem, then there exist optimal controls $\boldsymbol{\tau}^*(t)$, optimal transition times $t_i^* \in \mathcal{I}_T$, dual multipliers $\boldsymbol{\xi}_i^{+*}, \boldsymbol{\xi}_i^{-*}$, in $\mathbb{R}^{1 \times n}$, $\forall t_i^* \in \mathcal{I}_T$, transition location triplets $\{\mathbf{q}^*(t_i), \mathbf{u}^*(t_i^+), \mathbf{u}^*(t_i^-)\}$, dual state $\boldsymbol{\nu}^*(t^-) \in \mathcal{LCLBV}^*[\mathbb{R}^{1 \times n}]$ (where $*$ denote dual space) and a scalar $\lambda(t^+) \in \{0, 1\}$, such that $\lambda(t^+) + |\boldsymbol{\nu}^*(t^-)| > 0$ for all $t \in \Omega_t \cup \mathcal{I}_T$, which fulfill:

1. the Lebesgue-measurable dynamics in every interval of motion $t \in (t_i^{+*}, t_{i+1}^{-*})$

$$\mathbf{M}(\mathbf{q}^*)\dot{\mathbf{u}}^* - \mathbf{h}_i(\mathbf{q}^*, \mathbf{u}^*) - \mathbf{B}_i(\mathbf{q}^*) \boldsymbol{\tau}^* = \mathbf{0} \quad \text{a.e.}, \quad (4.160)$$

2. the Lebesgue-measurable dual dynamics on every interval (t_i^{+*}, t_{i+1}^{-*})

$$\ddot{\boldsymbol{\nu}}^*(t^-) \mathbf{D}_i + \dot{\boldsymbol{\nu}}^*(t^-) \mathbf{E}_i + \boldsymbol{\nu}^*(t^-) \mathbf{F}_i = \mathbf{0} \quad \text{a.e.}, \quad (4.161)$$

where the coefficients in the differential equation above are given by:

$$\begin{aligned} \mathbf{D}_i &= \mathbf{M}(\mathbf{q}^*), \\ \mathbf{E}_i &= 2\dot{\mathbf{M}}(\mathbf{q}^*, \mathbf{u}^*) + \nabla_{\mathbf{u}} \mathbf{h}_i(\mathbf{q}^*, \mathbf{u}^*), \\ \mathbf{F}_i &= \\ &\nabla_{\mathbf{q}} [\mathbf{M}(\mathbf{q}^*) \dot{\mathbf{u}}^* - \mathbf{h}_i(\mathbf{q}^*, \mathbf{u}^*) - \mathbf{B}_i(\mathbf{q}^*) \boldsymbol{\tau}^*] + \ddot{\mathbf{M}}(\mathbf{q}^*, \mathbf{u}^*, \dot{\mathbf{u}}^*) + \frac{d}{dt} (\nabla_{\mathbf{u}} [\mathbf{h}_i(\mathbf{q}^*, \mathbf{u}^*)]), \end{aligned}$$

3. the optimal control law on every interval (t_i^{+*}, t_{i+1}^{-*})

$$\boldsymbol{\nu}^*(t^-) \mathbf{B}_i(\mathbf{q}^*(t^+)) \in \mathcal{N}_{\mathcal{C}_\tau}(\boldsymbol{\tau}^{+*}), \quad \text{a.e.}, \quad (4.162)$$

4. The discontinuities in the dual state $\boldsymbol{\nu}^*(t^-)$ and its time derivative $\dot{\boldsymbol{\nu}}^*(t^-)$ are governed for all $t_i^* \in \mathcal{I}_T$ by:

$$\boldsymbol{\Omega}_i^{-*} - \boldsymbol{\Omega}_i^{+*} = -\boldsymbol{\xi}_i^* \mathbf{V},$$

where $\boldsymbol{\xi}_i^* = \boldsymbol{\xi}_i^{+*} + \boldsymbol{\xi}_i^{-*}$, and

$$\mathbf{V} = \begin{bmatrix} \nabla_{\mathbf{q}} (\mathbf{W}_b^T \mathbf{u}(t_i^+)) \\ \nabla_{\mathbf{q}} (\mathbf{W}_f^T \mathbf{u}(t_i^+)) - \nabla_{\mathbf{q}} (K \mathbf{u}(t_i^-)) \end{bmatrix}. \quad (4.163)$$

Further,

$$(\boldsymbol{\nu}^*(t_i^+) - \boldsymbol{\nu}^*(t_i^-)) \mathbf{M}(\mathbf{q}(t)) = \boldsymbol{\xi}_i^* \begin{bmatrix} \mathbf{W}_b^T \\ \mathbf{G}_{fb} \mathbf{G}_{bb}^{-1} \mathbf{W}_b^T \end{bmatrix}, \quad (4.164)$$

for some $\boldsymbol{\xi}_i^{+*}, \boldsymbol{\xi}_i^{-*} \in \mathbb{R}^n$ is necessary,

5. The jump in the generalised velocities shall fulfill:

$$\mathbf{u}^*(t_i^+) = \mathbf{P}_\perp^\top(\mathbf{q}^*(t_i)) \mathbf{u}^*(t_i^-), \quad \forall t_i^* \in \mathcal{I}_T. \quad (4.165)$$

and

$$\mathcal{C}_{T_i}^* \cap \mathcal{C}_{I_i}^* = \left\{ \left[\begin{array}{c} \mathbf{q}^*(t_i) \\ \mathbf{u}^*(t_i^-) \\ \mathbf{u}^*(t_i^+) \end{array} \right] \middle| \mathbf{L}(\mathbf{q}^*(t_i)) \begin{bmatrix} \mathbf{u}^*(t_i^+) \\ \mathbf{u}^*(t_i^-) \end{bmatrix} = \mathbf{0} \right\}, \quad \forall t_i^* \in \mathcal{I}_T.$$

with

$$\mathbf{L}(\mathbf{q}^*(t_i)) = \begin{bmatrix} \mathbf{W}_b^T(\mathbf{q}^*(t_i)) & \mathbf{0} \\ \mathbf{W}_f^T(\mathbf{q}^*(t_i)) & -\mathbf{K}(\mathbf{q}^*(t_i)) \end{bmatrix}, \quad (4.166)$$

6. the condition for all $t_i^* \in \mathcal{I}_T$

$$\Upsilon_i^{+*} + \Upsilon_i^{-*} = \mathbf{r}_{i1} \mathbf{u}^*(t_i^-) + \mathbf{r}_{i2} \mathbf{u}^*(t_i^+) + \mathbf{r}_{i3} \dot{\mathbf{u}}^*(t_i^-) + \mathbf{r}_{i4} \dot{\mathbf{u}}^*(t_i^+),$$

where the vectors $\mathbf{r}_{i1}$, $\mathbf{r}_{i2}$, $\mathbf{r}_{i3}$ and $\mathbf{r}_{i4}$ are given by:

$$\begin{aligned} \mathbf{r}_1 \mathbf{u}^*(t_i^-) + \mathbf{r}_2 \mathbf{u}^*(t_i^+) &= \mathbf{V} (\boldsymbol{\xi}_i^{-*} \mathbf{u}^*(t_i^-) + \boldsymbol{\xi}_i^{+*} \mathbf{u}^*(t_i^+)) . \\ \mathbf{r}_3 &= -\boldsymbol{\xi}_i^{-*} \begin{bmatrix} \mathbf{0} \\ -\mathbf{K} \end{bmatrix}, \quad \mathbf{r}_4 = -\boldsymbol{\xi}_i^{+*} \begin{bmatrix} \mathbf{W}_b^T \\ \mathbf{W}_f^T \end{bmatrix}, \end{aligned}$$

7. the boundary constraints $\mathcal{C}_f$,

8. the variational inequality with respect to the variations at final time $\hat{t}_f$:

$$\left(\Upsilon_f^* - \langle \Psi_{\mathcal{C}_f}^\uparrow(\cdot; \hat{\mathbf{q}}(t_f)), \mathbf{u}^*(t_f) \rangle - \langle \Psi_{\mathcal{C}_f}^\uparrow(\cdot; \hat{\mathbf{u}}(t_f)), \dot{\mathbf{u}}^*(t_f) \rangle \right) \hat{t}_f \geq 0, \quad \forall \hat{t}_f \in \mathbb{R}, \quad (4.167)$$

9. the transversality condition at final state:

$$\begin{bmatrix} -\boldsymbol{\Omega}_f^* \\ -\boldsymbol{\nu}^*(t_f) \mathbf{M} \end{bmatrix} \in \mathcal{N}_{\mathcal{C}_f}(\mathbf{q}^*(t_f), \mathbf{u}^*(t_f)). \quad (4.168)$$

Here $\boldsymbol{\Omega}_i$ and $\boldsymbol{\Upsilon}_i$ are defined by

$$\begin{aligned} \boldsymbol{\Omega}_i &= -(\dot{\boldsymbol{\nu}}(t_i) \mathbf{M} + \boldsymbol{\nu}(t_i) \dot{\mathbf{M}}) - \boldsymbol{\nu}(t_i) \nabla_{\mathbf{u}(t_i)} \mathbf{h}_i, \\ \boldsymbol{\Upsilon}_i &= -\boldsymbol{\nu}(t_i) (\nabla_{\mathbf{u}} \mathbf{h}_i \mathbf{u}(t_i) + \mathbf{M} \dot{\mathbf{u}}(t_i)) - \left(\dot{\boldsymbol{\nu}}(t_i) \mathbf{M} + \boldsymbol{\nu}(t_i) \dot{\mathbf{M}} \right) \mathbf{u}(t_i). \end{aligned}$$

Chapter 5

Necessary Conditions in the Hamiltonian Framework

The aim of this section is to derive an impulsive PMP by making use of the internal boundary variations and discontinuous transversality conditions for finite-dimensional Lagrangian systems. By the application of subdifferential calculus techniques to extended-valued lower semi-continuous functionals, Pontryagin's Maximum Principle (PMP) like conditions are obtained. The considered functional is a generalized Bolza functional that is evaluated on multiple intervals. The well-known PMP entails the necessary conditions for optimal control problems with differential constraints and end-point constraints with sufficient regularity properties in the space of absolutely continuous arcs ($\mathcal{AC}$). However, impulsive optimal control requires to search extremizing arcs in the space of bounded variation arcs ($\mathcal{BV}$). So the obtained necessary conditions encompasses PMP conditions under mild hypotheses since the class of $\mathcal{BV}$ arcs totally encompass the class of $\mathcal{AC}$ arcs. For the introduction to variational calculus with elementary convexity considerations, the monograph of Troutman [101] recommended. For an advanced introduction to the theory of subgradients and its applications to optimisation in the nonconvex setting the monograph [88] is didactically suitable. A very good overview to the application of nonsmooth analysis to control theory is given in the monograph [31]. The monograph of Vinter [103] presents the up to date state of optimal control in the nonsmooth setting. The main result in this chapter is presented and published in [110] by Yunt.

5.1 Derivation of Necessary Conditions in first-order form

In this section the necessary conditions for the optimal control problem of a structure-variant mechanical system are derived. The specific problem considered has an impactive transition at time t_i , such that the structure of the equations of motion in the pre-transition and post-transition phases may differ. In order to deal with the change of dimension of admissible

variations, the projective approach is used as discussed in subsection 2.6.2. A first-order representation of mechanical dynamics is used in order to interpret the characteristics of the investigated optimal control problem in the hamiltonian framework.

5.2 Generalised Problem of Bolza

Let us consider a problem in Bolza form (*GPB*), in which the objective is to choose an absolutely continuous arc $\mathbf{x} \in \mathcal{AC}$ in order to minimize

$$P: \quad J(\mathbf{x}) = l(\mathbf{x}(a), \mathbf{x}(b)) + \int_a^b L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) dt \quad (5.1)$$

where the function $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is $\mathcal{L} \times \mathcal{B}$ measurable. Here $\mathcal{L} \times \mathcal{B}$ denotes the σ -algebra of subsets of $[a, b] \times \mathbb{R}^n$ generated by product sets $\mathcal{M} \times \mathcal{N}$, where $\mathcal{M}$ is a Lebesgue measurable subset of $[a, b]$ and $\mathcal{N}$ is a Borel subset of $\mathbb{R}^{2n}$. For each $t \in [a, b]$, the function l and L are lower semi-continuous (l.s.c) on $\mathbb{R}^n \times \mathbb{R}^n$, with values in $\mathbb{R} \cup \{+\infty\}$. For each $(t, \mathbf{x})$ in $[a, b] \times \mathbb{R}^n$, the function $L(t, \mathbf{x}, \cdot)$ is convex and l represents the endpoint cost. GPB concerns the minimisation of a functional whose form is identical to that in the classical calculus of variations. The GPB is distinguished from its classical version, by the extremely mild hypotheses imposed on the endpoint cost l and the integrand L . Both are allowed to take the value $+\infty$. An important class of optimal control problems constrain the derivative of an admissible arc and they can be stated as the following Mayer problem:

$$\min\{l(\mathbf{x}(a), \mathbf{x}(b)) : \dot{\mathbf{x}}(t) \in \mathcal{F}(t, \mathbf{x}(t)), \quad \text{a.e. } t \in [a, b]\}. \quad (5.2)$$

The Mayer problem can be seen as minimizing the Bolza functional J over all arc $\mathbf{x}$. To cover the Mayer problem, it suffices to choose:

$$L(t, \mathbf{x}, \mathbf{v}) = \Psi_{\mathcal{F}(t, \mathbf{x})}(\mathbf{v}) = \begin{cases} 0, & \text{if } \mathbf{v} \in \mathcal{F}(t, \mathbf{x}) \\ +\infty, & \text{otherwise.} \end{cases} \quad (5.3)$$

The function $\Psi_{\mathcal{C}}$ is called the indicator function of the set $\mathcal{C}$. It is evident that for any arc $\mathbf{x}$, one has

$$\int_a^b L(t, \mathbf{x}, \dot{\mathbf{x}}) dt = \begin{cases} 0, & \text{if } \dot{\mathbf{x}}(t) \in \mathcal{F}(t, \mathbf{x}) \quad \text{a.e.} \\ +\infty, & \text{otherwise.} \end{cases} \quad (5.4)$$

The Mayer type variational problem can arise from a typical dynamic constraint in controls such as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \boldsymbol{\tau}(t)), \quad \text{a.e. } t \in [a, b], \quad \boldsymbol{\tau}(t) \in \mathcal{C}_{\tau}. \quad (5.5)$$

If a control state-pair $(\boldsymbol{\tau}, \mathbf{x})$ satisfies equation (5.5), then

$$\dot{\mathbf{x}}(t) \in \mathcal{F}(t, \mathbf{x}(t)) := \{\mathbf{f}(t, \mathbf{x}(t), \boldsymbol{\tau}(t)) : \boldsymbol{\tau}(t) \in \mathcal{C}_{\tau} \text{ a.e. } t \in [a, b]\} \quad (5.6)$$

certainly does. The well-known Fillipov's theorem is the statement that the reversal of the above statement is true.

In order to guarantee the well-behaving of $\mathcal{F}$ and l let following hypotheses hold:

Assumptions 5.2. An arc $\bar{\mathbf{x}} : [a, b] \rightarrow \mathbb{R}^n$ is given. On some relatively open subset $\Omega \subseteq [a, b] \times \mathbb{R}^n$ containing the graph of $\bar{\mathbf{x}}$, the following statements hold:

1. The multifunction $\mathcal{F}$ is $\mathcal{L} \times \mathcal{B}$ measurable on Ω . For each $(t, \mathbf{x})$ in Ω , the set $\mathcal{F}(t, \mathbf{x})$ is nonempty, compact and convex.
2. There are nonnegative integrable functions k and Φ on $[a, b]$ such that
 - (a) $\mathcal{F}(t, \mathbf{x}) \subseteq \Phi(t) \mathbb{B}$ for all $\mathbf{x}$ in Ω_t , almost everywhere, and
 - (b) $\mathcal{F}(t, \mathbf{x}) \subseteq \mathcal{F}(t, \mathbf{x}) + k(t)|\mathbf{y} - \mathbf{x}| \mathbb{B}$ for all $\mathbf{x}, \mathbf{y} \in \Omega_t$, almost everywhere.
3. The endpoint cost function l is l.s.c on $\Omega_a \times \Omega_b$.

where $\Omega_t = \{\mathbf{x} \in \mathbb{R}^n : (t, \mathbf{x}) \in \Omega\}$ for each t in $[a, b]$ and $\mathbb{B}$ is the unit open ball.

The generalised problem of many practical problems place constraints not only on the derivative of the state trajectory, but also on its endpoints. The differential inclusion problem is augmented with the additional constraint $(\mathbf{x}(a), \mathbf{x}(b)) \in \mathcal{S}$, where $\mathcal{S}$ is a given target set in $\mathbb{R}^n \times \mathbb{R}^n$ and is assumed to be closed. Suppose that there is a function $\varphi(t, \mathbf{x})$ with the following properties:

1. $\varphi(t, \mathbf{x}) \in \mathcal{F}(t, \mathbf{x})$ for all $\mathbf{x} \in \Omega_t$, almost everywhere;
2. $\varphi(t, \mathbf{x})$ is a Carathéodory function, i.e., φ is $\mathcal{L} \times \mathcal{B}$ measurable on Ω , and for almost every t the function $\mathbf{x} \mapsto \varphi(t, \mathbf{x})$ is Lipschitz on Ω_t with Lipschitz rank $k(t)$;
3. $\dot{\bar{\mathbf{x}}}(t) = \varphi(t, \bar{\mathbf{x}}(t))$ almost everywhere on $[a, b]$.

Given this setting the well-known PMP is stated in the formalism of nonsmooth analysis as follows:

Theorem 5.1 -Pontryagin's Maximum Principle Consider the optimal control problem of minimizing the endpoint function

$$l(\mathbf{x}(a), \mathbf{x}(b)) + \Psi_{\mathcal{S}}(\mathbf{x}(a), \mathbf{x}(b)) \quad (5.7)$$

over all arcs $\mathbf{x}$ satisfying the differential constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \boldsymbol{\tau}(t)), \quad \boldsymbol{\tau}(t) \in \mathcal{C}_{\tau}, \quad \text{a.e. } t \in [a, b]. \quad (5.8)$$

In addition, suppose that f is a Carathéodory function for which the velocity sets $\mathcal{F}(t, \mathbf{x}(t)) = \{\mathbf{v} | \mathbf{v} = \mathbf{f}(t, \mathbf{x}(t), \boldsymbol{\tau}(t)), \boldsymbol{\tau}(t) \in \mathcal{C}_{\tau}\}$ satisfy assumptions 5.2. If an arc $\bar{\mathbf{x}}$, together with a corresponding control function $\bar{\boldsymbol{\tau}}$, solves this problem, then there exist an arc $\mathbf{p} \in \mathcal{AC}$ on $[a, b]$ and a scalar λ equal to either 0 or 1 for which one has, for almost every $t \in [a, b]$,

- the adjoint equation,

$$-\dot{\mathbf{p}}(t) \in \bar{\partial}_x \langle \mathbf{p}(t), \mathbf{f}(t, \bar{\mathbf{x}}(t), \boldsymbol{\tau}(t)) \rangle, \quad (5.9)$$

- the maximum condition

$$\langle \mathbf{p}(t), \mathbf{f}(t, \bar{\mathbf{x}}, \bar{\boldsymbol{\tau}}) \rangle = \sup \{ \langle \mathbf{p}, \mathbf{f}(t, \bar{\mathbf{x}}, \boldsymbol{\tau}) \rangle : \boldsymbol{\tau} \in \mathcal{C}_\tau \}, \quad (5.10)$$

- the transversality condition

$$(\mathbf{p}(a), -\mathbf{p}(b)) \in \lambda \partial l(\bar{\mathbf{x}}(a), \bar{\mathbf{x}}(b)) + \mathcal{N}_S(\bar{\mathbf{x}}(a), \bar{\mathbf{x}}(b)). \quad (5.11)$$

Here $\mathcal{N}_S(\bar{\mathbf{x}}(a), \bar{\mathbf{x}}(b))$ denotes the limiting normal cone to the set $\mathcal{S}$ at $(\bar{\mathbf{x}}(a), \bar{\mathbf{x}}(b))$. The operator $\bar{\partial}$ denotes generalised subdifferential in the sense of Clarke. The above stated form of the Pontryagin's maximum principle (PMP) defines the necessary conditions for an arc $\bar{\mathbf{x}} \in \mathcal{AC}$ to extremize the Mayer problem subject to the constraints $\mathcal{S}$.

In the impulsive optimal control, it is assumed that the control horizon is composed of n different phases, which are separated from each other by $N - 1$ possibly discontinuous transitions. The importance of the transition process becomes clear if one considers the fact that at pre-transition and post-transition states the values of several functions may differ due to discontinuities. A transition process is common to the pre-transition configuration and post-transition configuration. Each problem P_i with a unique mechanical configuration is defined on a closed time domain $\text{dom}(P_i)$ with variable boundary which is partitioned as follows:

$$\text{dom}(P_i) = \{t_i^-, t_i^+\} \cup (t_i^+, t_{i+1}^-) \cup \{t_{i+1}^-, t_{i+1}^+\}. \quad (5.12)$$

The boundary of the domain $\text{dom}(P_i)$ is given by:

$$\text{bdy dom}(P_i) = \{t_i^-, t_i^+\} \cup \{t_{i+1}^-, t_{i+1}^+\}. \quad (5.13)$$

The interior of the domain is given by:

$$\text{int dom}(P_i) = (t_i^+, t_{i+1}^-). \quad (5.14)$$

The domain of the overall problem P_{Tot} is given by the union:

$$\text{dom}(P_{\text{Tot}}) = \bigcup_{\forall i} \text{dom}(P_i). \quad (5.15)$$

However, the domains of successive problems P_i and P_{i+1} are not disjoint:

$$\text{dom}(P_i) \cap \text{dom}(P_{i+1}) = \text{bdy dom}(P_i) \cap \text{bdy dom}(P_{i+1}) = \{t_{i+1}^-, t_{i+1}^+\}, \quad (5.16)$$

The set $\text{bdy dom}(P_i) \cap \text{bdy dom}(P_{i+1}) = \{t_{i+1}^-, t_{i+1}^+\}$ is the support of the transition process and is Lebesgue-negligible.

Having set the stage, the necessary conditions for the impulsive optimal control of structure-variant rigid-body mechanical systems is formally derived by considering a problem in Bolza form (*GPB*), in which the objective is to choose an arc $\mathbf{x} \in \mathcal{BV}$ in order to minimize

$$J(\mathbf{x}, \{\mathbf{x}(t_i^-), \mathbf{x}(t_i^+), t_i\}) = \sum_{i=1}^N l_i(\mathbf{x}(t_i^-), \mathbf{x}(t_i^+)) + \int_{t_i^+}^{t_{i+1}^-} L_i(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) dt. \quad (5.17)$$

The overall problem as stated in (5.17) is seen as the union of several problems in the generalised Bolza form. The theory at hand treats optimal solutions as solutions of multi-point boundary value problems (MBVP) with discontinuous transitions in the state. In this setting, the prespecification of the mode sequence and number of intervals must be given in advance. Here it is assumed that the control horizon is composed of n different phases, which are separated from each other by $N - 1$ possibly discontinuous transitions in the generalised velocities. The extended-valued integrand may differ on each interval based on the structure of the equations of motion. The difference in structure may arise due to change in parameters (i.e. mass, inertia) or degrees of freedom. In [113] by Yunt a projection approach is presented in case, the mechanical configurations in successive intervals differ based on change in the number of degrees of freedom.

5.3 Impulsive Optimal Control Problem in First-Order Form

The time-optimal control problem with free end-time t_f and free transition times t_i and location $\mathbf{q}(t_i)$, $\mathbf{u}(t_i^-)$, $\mathbf{u}(t_i^+)$ is considered. The assumptions during a possibly impactive transition are given as follows:

Assumptions 5.2

1. The transitions may be impactively.
2. The generalised position remain unchanged during transition.
3. The impulsive control action acts on the system at a time instant t_i which is Lebesgue-negligible and there are countably many of such transition instants.
4. At a possibly impactive transition, the pre-transition controller configuration is assumed to be effective.
5. There are no transitions at t_0 and t_f .

The necessary conditions are derived by making use of following assumptions on the general problem:

Assumptions 5.3

1. the dual states $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are assumed left-continuous locally bounded variation functions ($\mathcal{LCLBV}$), and the generalised velocities $\mathbf{u}$ of the Lagrangian system is assumed right-continuous locally bounded variation functions ($\mathcal{RCLBV}$), whereas the generalised positions are in class $\mathcal{AC}$.
2. The mode sequence and number of intervals for the MBVP constitute a feasible hybrid trajectory.
3. The set $\mathcal{C}_{I_i}^+ \cap \mathcal{C}_{T_i}^+$ is closed and nonempty.
4. The set $\mathcal{C}_{I_i}^- \cap \mathcal{C}_{T_i}^-$ is closed and nonempty.
5. The goal functional $g(\mathbf{q}, \mathbf{y}, \boldsymbol{\tau})$ is continuously differentiable for all $t \in \Omega_t$ and $t_i \in \mathcal{I}_T$.
6. The derivatives $\partial_{\mathbf{y}} g(\mathbf{q}, \mathbf{y}, \boldsymbol{\tau})$ are bounded for all $t \in \Omega_t$ and $t_i \in \mathcal{I}_T$.
7. Each $L_i : (t_i^+, t_{i+1}^-) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a Lebesgue normal integrand.
8. Each $L_i(\mathbf{q}(s), \mathbf{y}(s), \cdot)$ is convex for each $(\mathbf{q}(s), \mathbf{y}(s))$.
9. Each $l_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

The value function J has some pleasant regularity properties if assumptions 5.3 hold. Making use of these regularity properties, "sharp" necessary conditions are derived. The goal function is given by:

$$\min \int_{t_0}^{t_f} dt, \quad (5.18)$$

subject to the mechanical system dynamics stated in the first-order measure-differential equation form:

$$d\mathbf{q} = \mathbf{y} dt, \quad (5.19)$$

$$d\mathbf{y} = (\mathbf{f}_i(\mathbf{q}(t), \mathbf{y}(t)) + \mathbf{G}_i(\mathbf{q}(t))\boldsymbol{\tau}(t)) dt + \mathbf{V}_i(\mathbf{q}(t))\boldsymbol{\zeta}' d\sigma. \quad (5.20)$$

The smooth dynamics of the rigidbody mechanical system is characterised in every interval of motion (t_i^+, t_{i+1}^-) by a triplet $\{\mathbf{f}_i(\mathbf{q}(t), \mathbf{y}(t)), \mathbf{G}_i(\mathbf{q}(t)), \mathbf{V}_i(\mathbf{q}(t))\}$. By the Lebesgue-Stieltjes integration of equation (5.20) over an atom of time instant $t_i \in \mathcal{I}_T$ one obtains:

$$\int_{\{t_i\}} d\mathbf{y} - (\mathbf{f}_i + \mathbf{G}_i\boldsymbol{\tau}(t)) dt - \mathbf{V}_i\boldsymbol{\zeta}' d\sigma = \mathbf{y}(t_i^+) - \mathbf{y}(t_i^-) - \mathbf{V}_i(\mathbf{q}(t_i)) (\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) \quad (5.21)$$

which is an impact equation in first order form. In performing the integration in (5.21), it is assumed that the generalised impulsive control force directions $\mathbf{V}_i(\mathbf{q}(t_i))$ do not change their

structure. In performing the integration in (5.21), it is assumed that the generalised impulsive control force directions $\mathbf{V}_i(\mathbf{q}(t_i))$ do not change their structure. The vector controls $\boldsymbol{\tau}$ is assumed to be constrained in a polytopic compact convex set denoted by $\mathcal{C}_\tau$. The overall value functional is given by:

$$J = \Psi_{\mathcal{C}_f} + \sum_{i \in \mathcal{I}_T} \Psi_{\mathcal{C}_{T_i}^-} + \Psi_{\mathcal{C}_{T_i}^+} + \int_{(t_0, t_f)} \Psi_{\mathcal{C}_\tau} dt + dH - \langle \boldsymbol{\eta}_1, d\mathbf{q} \rangle - \langle \boldsymbol{\eta}_2, d\mathbf{y} \rangle. \quad (5.22)$$

The differential measure of the Hamiltonian is defined as:

$$dH = H_t dt + H_\sigma d\sigma = \langle \boldsymbol{\eta}_2(t), \mathbf{V}_i \boldsymbol{\zeta}' \rangle d\sigma + (\lambda(t) g(\mathbf{q}, \mathbf{y}, \boldsymbol{\tau}) + \langle \boldsymbol{\eta}_1(t), \mathbf{y}(t) \rangle + \langle \boldsymbol{\eta}_2(t), (\mathbf{f}_i + \mathbf{G}_i \boldsymbol{\tau}(t)) \rangle) dt \quad (5.23)$$

where $\boldsymbol{\eta}_1(t) \in \mathcal{LCBV}^*(\mathbb{R}^{1 \times n})$ and $\boldsymbol{\eta}_2(t) \in \mathcal{LCBV}^*(\mathbb{R}^{1 \times n})$ are the dual states. The unconstrained functional in (5.22) is equivalent to (5.24) under assumption 5.1.4:

$$J = \Psi_{\mathcal{C}_f} + \sum_{i \in \mathcal{I}_T} \Psi_{\mathcal{C}_{T_i}^-} + \Psi_{\mathcal{C}_{T_i}^+} + \Psi_{\mathcal{C}_{T_i}^-} + \Psi_{\mathcal{C}_{T_i}^+} + \int_{t_i^+}^{t_{i+1}^-} \Psi_{\mathcal{C}_\tau} + H_t - \langle \boldsymbol{\eta}_1, \mathbf{u} \rangle - \langle \boldsymbol{\eta}_2, \dot{\mathbf{y}} \rangle dt. \quad (5.24)$$

Following structure for various differential measures is noted:

$$\begin{aligned} d\mathbf{q} &= \mathbf{u} dt + \boldsymbol{\rho}' d\sigma, & d\mathbf{y} &= \ddot{\mathbf{q}} dt + \boldsymbol{\chi}' d\sigma, \\ d\boldsymbol{\eta}_1 &= \dot{\boldsymbol{\eta}}_1 dt + \boldsymbol{\xi}'_1 d\sigma, & d\boldsymbol{\eta}_2 &= \dot{\boldsymbol{\eta}}_2 dt + \boldsymbol{\xi}'_2 d\sigma. \end{aligned}$$

The vector controls $\boldsymbol{\tau}$ is assumed to be constrained in a polytopic convex set denoted by $\mathcal{C}_\tau$. Here the sets are defined as below:

$$\mathcal{T}_i^+ = \{ \{ \mathbf{q}(t_i^+), \mathbf{u}(t_i^-), \mathbf{u}(t_i^+) \} \mid \mathbf{A}_1(\mathbf{q}(t_i^+))\mathbf{u}(t_i^-) + \mathbf{A}_2(\mathbf{q}(t_i^+))\mathbf{u}(t_i^+) = \mathbf{0} \}, \quad (5.25)$$

$$\mathcal{T}_i^- = \{ \{ \mathbf{q}(t_i^-), \mathbf{u}(t_i^-), \mathbf{u}(t_i^+) \} \mid \mathbf{A}_1(\mathbf{q}(t_i^-))\mathbf{u}(t_i^-) + \mathbf{A}_2(\mathbf{q}(t_i^-))\mathbf{u}(t_i^+) = \mathbf{0} \}, \quad (5.26)$$

$$\mathcal{C}_f = \{ \{ \mathbf{q}(t_f), \mathbf{u}(t_f) \} \mid \mathbf{q}(t_f) = \mathbf{q}_f, \quad \mathbf{u}(t_f) = \mathbf{u}_f \}, \quad (5.27)$$

$$\mathcal{C}_\tau = \{ \boldsymbol{\tau} \mid \boldsymbol{\tau} \in \mathcal{K}, \text{convex, polytopic} \}, \quad (5.28)$$

$$\mathcal{C}_{T_i}^+ = \{ \{ \mathbf{q}(t_i^+), \mathbf{y}(t_i^+), \mathbf{y}(t_i^-) \} \mid \mathbf{y}(t_i^+) - \mathbf{y}(t_i^-) - \mathbf{V}_i(\mathbf{q}(t_i^+)) (\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) = \mathbf{0} \}, \quad (5.29)$$

$$\mathcal{C}_{T_i}^- = \{ \{ \mathbf{q}(t_i^-), \mathbf{y}(t_i^+), \mathbf{y}(t_i^-) \} \mid \mathbf{y}(t_i^+) - \mathbf{y}(t_i^-) - \mathbf{V}_i(\mathbf{q}(t_i^-)) (\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) = \mathbf{0} \}. \quad (5.30)$$

Here $\mathbf{A}_1$ and $\mathbf{A}_2$ are $n \times n$ dimensional linear operators. The set of transition conditions at each transition instant t_i are denoted by $\mathcal{T}_i$, and are stated in terms of generalised positions $\mathbf{q}(t_i)$, and generalised post-, and pre-transition velocities $\mathbf{u}(t_i^+)$, $\mathbf{u}(t_i^-)$.

The directional derivatives of the augmented cost function in equation (5.24) is evaluated with respect to $\hat{\mathbf{q}}(t)$, $\hat{\mathbf{y}}(t)$, $\hat{\dot{\mathbf{y}}}(t)$, $\hat{\boldsymbol{\tau}}(t)$, $\hat{\mathbf{q}}(t_f)$, $\hat{\mathbf{q}}(t_i^+)$, $\hat{\mathbf{q}}(t_i^-)$, $\hat{\mathbf{y}}(t_f)$, $\hat{\mathbf{y}}(t_i^+)$, $\hat{\mathbf{y}}(t_i^-)$, $\hat{t}_f$, $\hat{t}_i^+$, $\hat{t}_i^-$, $\hat{\boldsymbol{\zeta}}_i^+$, $\hat{\boldsymbol{\zeta}}_i^-$. The directional derivative in direction $\hat{\mathbf{q}}(t)$ is given by:

$$J^\uparrow(\cdot, \hat{\mathbf{q}}(t)) = \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \langle \nabla_{\mathbf{q}} H_t^{i+1}, \hat{\mathbf{q}}(t) \rangle dt, \quad \forall t \in \text{int dom}(P_i). \quad (5.31)$$

The directional derivative in direction $\hat{\mathbf{y}}(t)$ is given by:

$$J^\uparrow(\cdot, \hat{\mathbf{y}}(t)) = \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \langle \nabla_{\mathbf{y}} H_t^{i+1}, \hat{\mathbf{y}}(t) \rangle dt, \quad \forall t \in \text{int dom}(P_i). \quad (5.32)$$

The directional derivative in direction $\hat{\mathbf{u}}(t)$ is given by:

$$J^\uparrow(\cdot, \hat{\mathbf{u}}(t)) = - \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \langle \boldsymbol{\eta}_1(t), \hat{\mathbf{u}}(t) \rangle dt, \quad \forall t \in \text{int dom}(P_i). \quad (5.33)$$

The directional derivative in direction $\hat{\mathbf{y}}(t)$ is given by:

$$J^\uparrow(\cdot, \hat{\mathbf{y}}(t)) = - \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \langle \boldsymbol{\eta}_2(t), \hat{\mathbf{y}}(t) \rangle dt, \quad \forall t \in \text{int dom}(P_i). \quad (5.34)$$

The directional derivative in direction $\hat{\boldsymbol{\tau}}(t)$ is given by:

$$J^\uparrow(\cdot, \hat{\boldsymbol{\tau}}(t)) = \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \langle \nabla_{\boldsymbol{\tau}} H_t^{i+1}, \hat{\boldsymbol{\tau}}(t) \rangle + \Psi_{\mathcal{C}_\tau}^\uparrow(\cdot, \hat{\boldsymbol{\tau}}(t)) dt, \quad \forall t \in \text{int dom}(P_i). \quad (5.35)$$

The directional derivative in direction $\hat{t}_i^-$ at every $t_i \in \mathcal{I}_T$ is given by:

$$J^\uparrow(\cdot, \hat{t}_i^-) = (H_t^{i+1}(\mathbf{q}(t_i^-), \mathbf{y}(t_i^-)) - \langle \boldsymbol{\eta}_1(t_i^-), \mathbf{u}(t_i^-) \rangle - \langle \boldsymbol{\eta}_2(t_i^-), \dot{\mathbf{y}}(t_i^-) \rangle) \hat{t}_i^-. \quad (5.36)$$

The directional derivative in direction $\hat{t}_i^+$ at every $t_i \in \mathcal{I}_T$ is given by:

$$J^\uparrow(\cdot, \hat{t}_i^+) = - (H_t^{i+1}(\mathbf{q}(t_i^+), \mathbf{y}(t_i^+)) - \langle \boldsymbol{\eta}_1(t_i^+), \mathbf{u}(t_i^+) \rangle - \langle \boldsymbol{\eta}_2(t_i^+), \dot{\mathbf{y}}(t_i^+) \rangle) \hat{t}_i^+. \quad (5.37)$$

The directional derivative in direction $\hat{t}_f$ at final time t_f is given by:

$$J^\uparrow(\cdot, \hat{t}_f) = (H_t^i(\mathbf{q}(t_f), \mathbf{y}(t_f)) - \langle \boldsymbol{\eta}_1(t_f), \mathbf{u}(t_f) \rangle - \langle \boldsymbol{\eta}_2(t_f), \dot{\mathbf{y}}(t_f) \rangle) \hat{t}_f + \Psi_{\mathcal{C}_f}(\cdot, \hat{t}_f). \quad (5.38)$$

The directional derivatives of J in directions $\hat{\mathbf{q}}(t_f)$, $\hat{\mathbf{q}}(t_i^+)$, $\hat{\mathbf{q}}(t_i^-)$, $\hat{\mathbf{y}}(t_f)$, $\hat{\mathbf{y}}(t_i^+)$, $\hat{\mathbf{y}}(t_i^-)$, $\hat{\boldsymbol{\zeta}}_i^-$ and $\hat{\boldsymbol{\zeta}}_i^+$ at every $t_i \in \mathcal{I}_T$ are given by:

$$J^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^+)) = \Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^+)) + \Psi_{\mathcal{T}_i^+}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^+)), \quad (5.39)$$

$$J^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^-)) = \Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^-)) + \Psi_{\mathcal{T}_i^-}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^-)), \quad (5.40)$$

$$J^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)) = \Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)) + \Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)) + \Psi_{\mathcal{T}_i^-}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)) + \Psi_{\mathcal{T}_i^+}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)), \quad (5.41)$$

$$J^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)) = \Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)) + \Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)) + \Psi_{\mathcal{T}_i^-}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)) + \Psi_{\mathcal{T}_i^+}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)), \quad (5.42)$$

$$J^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^+) = \Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^+) + \Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^+), \quad (5.43)$$

$$J^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^-) = \Psi_{\mathcal{C}_i^+}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^-) + \Psi_{\mathcal{C}_i^-}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^-), \quad (5.44)$$

$$J^\uparrow(\cdot, \hat{\mathbf{q}}(t_f)) = \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{\mathbf{q}}(t_f)), \quad (5.45)$$

$$J^\uparrow(\cdot, \hat{\mathbf{y}}(t_f)) = \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{\mathbf{y}}(t_f)), \quad (5.46)$$

respectively. The directional derivatives of $\Psi_{C_i^+}$ in directions $\hat{\mathbf{q}}(t_i^+)$, $\hat{\mathbf{y}}(t_i^+)$, $\hat{\mathbf{y}}(t_i^-)$, $\hat{\boldsymbol{\zeta}}_i^-$ and $\hat{\boldsymbol{\zeta}}_i^+$ at every $t_i \in \mathcal{I}_T$ are given by:

$$\Psi_{C_i^+}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)) = -\langle \boldsymbol{\lambda}_i^+, \hat{\mathbf{y}}(t_i^-) \rangle, \quad (5.47)$$

$$\Psi_{C_i^+}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^+)) = \langle \boldsymbol{\lambda}_i^+, \nabla_{\mathbf{q}} \mathbf{V}_i(\mathbf{q}(t_i^+)) (\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) \hat{\mathbf{q}}(t_i^+) \rangle, \quad (5.48)$$

$$\Psi_{C_i^+}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)) = \langle \boldsymbol{\lambda}_i^+, \hat{\mathbf{y}}(t_i^+) \rangle, \quad (5.49)$$

$$\Psi_{C_i^+}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^+) = \langle \boldsymbol{\lambda}_i^+, \mathbf{V}_i(\mathbf{q}(t_i^+)) \hat{\boldsymbol{\zeta}}_i^+ \rangle, \quad (5.50)$$

$$\Psi_{C_i^+}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^-) = -\langle \boldsymbol{\lambda}_i^+, \mathbf{V}_i(\mathbf{q}(t_i^+)) \hat{\boldsymbol{\zeta}}_i^- \rangle, \quad (5.51)$$

respectively. The directional derivatives of $\Psi_{C_i^-}$ in directions $\hat{\mathbf{q}}(t_i^-)$, $\hat{\mathbf{y}}(t_i^+)$, $\hat{\mathbf{y}}(t_i^-)$, $\hat{\boldsymbol{\zeta}}_i^-$ and $\hat{\boldsymbol{\zeta}}_i^+$ at every $t_i \in \mathcal{I}_T$ are given by:

$$\Psi_{C_i^-}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^-)) = \langle \boldsymbol{\lambda}_i^-, \nabla_{\mathbf{q}} \mathbf{V}_i(\mathbf{q}(t_i^-)) (\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) \hat{\mathbf{q}}(t_i^-) \rangle, \quad (5.52)$$

$$\Psi_{C_i^-}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)) = \langle -\boldsymbol{\lambda}_i^-, \hat{\mathbf{y}}(t_i^-) \rangle, \quad (5.53)$$

$$\Psi_{C_i^-}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)) = \langle \boldsymbol{\lambda}_i^-, \hat{\mathbf{y}}(t_i^+) \rangle, \quad (5.54)$$

$$\Psi_{C_i^-}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^+) = \langle \boldsymbol{\lambda}_i^-, \mathbf{V}_i(\mathbf{q}(t_i^-)) \hat{\boldsymbol{\zeta}}_i^+ \rangle, \quad (5.55)$$

$$\Psi_{C_i^-}^\uparrow(\cdot, \hat{\boldsymbol{\zeta}}_i^-) = \langle -\boldsymbol{\lambda}_i^-, \mathbf{V}_i(\mathbf{q}(t_i^-)) \hat{\boldsymbol{\zeta}}_i^- \rangle, \quad (5.56)$$

respectively. The directional derivatives of $\Psi_{\mathcal{T}_i^+}$ in directions $\hat{\mathbf{q}}(t_i^+)$, $\hat{\mathbf{y}}(t_i^+)$ and $\hat{\mathbf{y}}(t_i^-)$ at every $t_i \in \mathcal{I}_T$ are given by:

$$\Psi_{\mathcal{T}_i^+}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)) = \langle \boldsymbol{\xi}_i^+, \mathbf{A}_1(\mathbf{q}(t_i^+)) \hat{\mathbf{y}}(t_i^-) \rangle, \quad (5.57)$$

$$\Psi_{\mathcal{T}_i^+}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^+)) = \langle \boldsymbol{\xi}_i^+, (\nabla_{\mathbf{q}} \mathbf{A}_1(\mathbf{q}(t_i^+)) \mathbf{y}(t_i^-) + \nabla_{\mathbf{q}} \mathbf{A}_2(\mathbf{q}(t_i^+)) \mathbf{y}(t_i^+)) \hat{\mathbf{q}}(t_i^+) \rangle, \quad (5.58)$$

$$\Psi_{\mathcal{T}_i^+}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)) = \langle \boldsymbol{\xi}_i^+, \mathbf{A}_2(\mathbf{q}(t_i^+)) \hat{\mathbf{y}}(t_i^+) \rangle. \quad (5.59)$$

The directional derivatives of $\Psi_{\mathcal{T}_i^-}$ in directions $\hat{\mathbf{q}}(t_i^-)$, $\hat{\mathbf{y}}(t_i^+)$ and $\hat{\mathbf{y}}(t_i^-)$ at every $t_i \in \mathcal{I}_T$ are given by:

$$\Psi_{\mathcal{T}_i^-}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^-)) = \langle \boldsymbol{\xi}_i^-, \mathbf{A}_1(\mathbf{q}(t_i^-)) \hat{\mathbf{y}}(t_i^-) \rangle, \quad (5.60)$$

$$\Psi_{\mathcal{T}_i^-}^\uparrow(\cdot, \hat{\mathbf{q}}(t_i^-)) = \langle \boldsymbol{\xi}_i^-, (\nabla_{\mathbf{q}} \mathbf{A}_1(\mathbf{q}(t_i^-)) \mathbf{y}(t_i^-) + \nabla_{\mathbf{q}} \mathbf{A}_2(\mathbf{q}(t_i^-)) \mathbf{y}(t_i^+)) \hat{\mathbf{q}}(t_i^-) \rangle, \quad (5.61)$$

$$\Psi_{\mathcal{T}_i^-}^\uparrow(\cdot, \hat{\mathbf{y}}(t_i^+)) = \langle \boldsymbol{\xi}_i^-, \mathbf{A}_2(\mathbf{q}(t_i^-)) \hat{\mathbf{y}}(t_i^+) \rangle. \quad (5.62)$$

respectively.

The dual vectors $\boldsymbol{\lambda}_i^+$, $\boldsymbol{\lambda}_i^-$, $\boldsymbol{\xi}_i^+$ and $\boldsymbol{\xi}_i^-$ are elements of $\mathbb{R}^n$. By using the Lemma of Raymond-Dubois in conjunction with proposition 2.1 the forms of equations (5.33) and (5.34) reveal on

very domain $\text{dom}(P_i)$:

$$-\int_{t_i^+}^{t_{i+1}^-} \langle \boldsymbol{\eta}_1(t), \hat{\mathbf{u}}(t) \rangle dt = -\langle \boldsymbol{\eta}_1(t), \hat{\mathbf{q}}(t) \rangle \Big|_{t_i^+}^{t_{i+1}^-} + \int_{t_i^+}^{t_{i+1}^-} \langle \dot{\boldsymbol{\eta}}_1(t), \hat{\mathbf{q}}(t) \rangle dt, \quad (5.63)$$

$$-\int_{t_i^+}^{t_{i+1}^-} \langle \boldsymbol{\eta}_2(t), \hat{\mathbf{y}}(t) \rangle dt = -\langle \boldsymbol{\eta}_2(t), \hat{\mathbf{y}}(t) \rangle \Big|_{t_i^+}^{t_{i+1}^-} + \int_{t_i^+}^{t_{i+1}^-} \langle \dot{\boldsymbol{\eta}}_2(t), \hat{\mathbf{y}}(t) \rangle dt, \quad (5.64)$$

respectively. After the integrations the directional derivatives in direction $\hat{\mathbf{q}}(t)$ and $\hat{\mathbf{y}}(t)$ can be combined as in equation (5.65):

$$J^\dagger(\cdot, \hat{\mathbf{q}}(t)) = \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \langle (\nabla_{\mathbf{q}} H_t^{i+1} + \dot{\boldsymbol{\eta}}_1(t)), \hat{\mathbf{q}}(t) \rangle dt. \quad (5.65)$$

Analogously, the directional derivatives in direction $\hat{\mathbf{y}}(t)$ and $\hat{\mathbf{y}}(t)$ can be combined as in equation (5.66):

$$J^\dagger(\cdot, \hat{\mathbf{y}}(t)) = \sum_{i=0}^{N-1} \int_{t_i^+}^{t_{i+1}^-} \langle (\nabla_{\mathbf{y}} H_t^{i+1} + \dot{\boldsymbol{\eta}}_2(t)), \hat{\mathbf{y}}(t) \rangle dt. \quad (5.66)$$

Further, the terms obtained by the directional derivatives $\hat{\mathbf{q}}(t_f)$, $\hat{\mathbf{y}}(t_f)$, $\hat{\mathbf{q}}(t_i^+)$, $\hat{\mathbf{y}}(t_i^+)$, $\hat{\mathbf{q}}(t_i^-)$, $\hat{\mathbf{y}}(t_i^-)$ can be brought in relation with the total variations $\hat{\mathbf{q}}_f$, $\hat{\mathbf{y}}_f$, $\hat{\mathbf{q}}_i^+$, $\hat{\mathbf{y}}_i^+$, $\hat{\mathbf{q}}_i^-$, $\hat{\mathbf{y}}_i^-$ by the following affine relations:

$$\hat{\mathbf{q}}(t_f) = \hat{\mathbf{q}}_f - \mathbf{u}(t_f) \hat{t}_f, \quad (5.67)$$

$$\hat{\mathbf{y}}(t_f) = \hat{\mathbf{y}}_f - \dot{\mathbf{y}}(t_f) \hat{t}_f, \quad (5.68)$$

$$\hat{\mathbf{q}}(t_i^+) = \hat{\mathbf{q}}_i^+ - \mathbf{u}(t_i^+) \hat{t}_i^+, \quad (5.69)$$

$$\hat{\mathbf{y}}(t_i^+) = \hat{\mathbf{y}}_i^+ - \dot{\mathbf{y}}(t_i^+) \hat{t}_i^+ - \hat{\boldsymbol{\chi}}_i^+, \quad (5.70)$$

$$\hat{\mathbf{q}}(t_i^-) = \hat{\mathbf{q}}_i^- - \mathbf{u}(t_i^-) \hat{t}_i^-, \quad (5.71)$$

$$\hat{\mathbf{y}}(t_i^-) = \hat{\mathbf{y}}_i^- - \dot{\mathbf{y}}(t_i^-) \hat{t}_i^- - \hat{\boldsymbol{\chi}}_i^-, \quad (5.72)$$

By making use of affine relations (5.67) to (5.72) the boundary variations are decomposed into orthogonal independent variations in $\hat{t}_i^-$, $\hat{t}_i^+$, $\hat{\mathbf{y}}_i^+$, $\hat{\mathbf{y}}_i^-$, $\hat{\mathbf{q}}_i^+$, $\hat{\mathbf{q}}_i^-$, $\hat{\boldsymbol{\zeta}}_i^-$, $\hat{\boldsymbol{\zeta}}_i^+$ at each transition instant. Thus the internal boundary variations at each transition time are given in the finite-dimensional set $\hat{\mathcal{V}}$:

$$\begin{aligned} \hat{\mathcal{V}} &= \left\{ \hat{t}_i^-, \hat{t}_i^+, \hat{\mathbf{y}}_i^+, \hat{\mathbf{y}}_i^-, \hat{\mathbf{q}}_i^+, \hat{\mathbf{q}}_i^-, \hat{\boldsymbol{\zeta}}_i^+, \hat{\boldsymbol{\zeta}}_i^-, \hat{\boldsymbol{\chi}}_i^-, \hat{\boldsymbol{\chi}}_i^+ \right\} \\ &\subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^n. \end{aligned} \quad (5.73)$$

The the internal boundary variations evaluated at transition time $t_i \in \mathcal{I}_T$ when recombined with the forms obtained in (5.63) and (5.64) become:

$$\langle \mathbf{a}_1, \hat{\mathbf{q}}_i^+ \rangle + \langle \mathbf{a}_2, \hat{\mathbf{q}}_i^- \rangle + \langle \mathbf{a}_3, \hat{\mathbf{y}}_i^+ \rangle + \langle \mathbf{a}_4, \hat{\mathbf{y}}_i^- \rangle + \langle \mathbf{a}_5, \hat{t}_i^+ \rangle + \langle \mathbf{a}_6, \hat{t}_i^- \rangle + \langle \mathbf{a}_7, \hat{\boldsymbol{\chi}}_i^+ \rangle + \langle \mathbf{a}_8, \hat{\boldsymbol{\chi}}_i^- \rangle, \quad (5.74)$$

where the vectors $\mathbf{a}_j$ for $j = 1, \dots, 8$ are given by:

$$\mathbf{a}_1 = \boldsymbol{\eta}_1(t_i^+), \quad \mathbf{a}_5 = -\boldsymbol{\eta}_1(t_i^+)\mathbf{y}(t_i^+) - \boldsymbol{\eta}_2(t_i^+)\dot{\mathbf{y}}(t_i^+), \quad (5.75)$$

$$\mathbf{a}_2 = -\boldsymbol{\eta}_1(t_i^-), \quad \mathbf{a}_6 = \boldsymbol{\eta}_1(t_i^-)\mathbf{y}(t_i^-) + \boldsymbol{\eta}_2(t_i^-)\dot{\mathbf{y}}(t_i^-), \quad (5.76)$$

$$\mathbf{a}_3 = \boldsymbol{\eta}_2(t_i^+), \quad \mathbf{a}_7 = \boldsymbol{\eta}_2(t_i^-), \quad (5.77)$$

$$\mathbf{a}_4 = -\boldsymbol{\eta}_2(t_i^-), \quad \mathbf{a}_8 = -\boldsymbol{\eta}_2(t_i^+). \quad (5.78)$$

By evaluating the terms associated with time variations in (5.74) with the terms given in (5.36) and (5.37) the total directional derivatives in the directions of $\hat{t}_i^-$, $\hat{t}_i^+$ are obtained by inserting the resulting terms in the relations of the directional derivatives at every $t_i \in \mathcal{I}_T$:

$$\begin{aligned} J^\uparrow(\cdot, \hat{t}_i^-) = & \quad (5.79) \\ & [H_t^i + \nabla_{\mathbf{q}} [-\boldsymbol{\lambda}_i^- \mathbf{V}_i(\mathbf{q}(t_i^+))\boldsymbol{\Gamma} - \boldsymbol{\xi}_i^- (\mathbf{A}_1(\mathbf{q}(t_i^-))\mathbf{y}(t_i^-) + \mathbf{A}_2(\mathbf{q}(t_i^-)), \mathbf{y}(t_i^+))]] \mathbf{y}(t_i^-) \\ & + [\boldsymbol{\lambda}_i^- - \boldsymbol{\xi}_i \mathbf{A}_1(\mathbf{q})] \dot{\mathbf{y}}(t_i^-) \hat{t}_i^-, \end{aligned}$$

and

$$\begin{aligned} J^\uparrow(\cdot, \hat{t}_i^+) = & \quad (5.80) \\ & [-H_t^{i+1} + \nabla_{\mathbf{q}} [-\boldsymbol{\lambda}_i^+ \mathbf{V}_i(\mathbf{q}(t_i^+))\boldsymbol{\Gamma} - \boldsymbol{\xi}_i^+ (\mathbf{A}_1(\mathbf{q}(t_i^+))\mathbf{y}(t_i^-) + \mathbf{A}_2(\mathbf{q}(t_i^+)), \mathbf{y}(t_i^+))]] \mathbf{y}(t_i^+) \\ & + [-\boldsymbol{\lambda}_i^+ - \boldsymbol{\xi}_i \mathbf{A}_1(\mathbf{q})] \dot{\mathbf{y}}(t_i^+) \hat{t}_i^+. \end{aligned}$$

By the assumption that transition times t_i are Lebesgue-negligible, the variations should be such that there occurs no gap on the time line. As a consequence the directional derivatives $J^\uparrow(\cdot, \hat{t}_i^-)$ and $J^\uparrow(\cdot, \hat{t}_i^+)$ can be seen as collinear and equivalent. Combining equations (5.79) and (5.80) following variational inequality for every $t_i \in \mathcal{I}_T$ is obtained:

$$\begin{aligned} J^\uparrow(\cdot, \hat{t}_i) = [H_t^i - H_t^{i+1} + \langle \mathbf{p}_1, \mathbf{y}(t_i^+) \rangle + \langle \mathbf{p}_2, \mathbf{y}(t_i^-) \rangle + \langle \mathbf{p}_3, \dot{\mathbf{y}}(t_i^+) \rangle + \langle \mathbf{p}_4, \dot{\mathbf{y}}(t_i^-) \rangle] \hat{t}_i \geq 0, \\ \forall \hat{t}_i \in \mathbb{R}, \end{aligned} \quad (5.81)$$

where the vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 \in \mathbb{R}^n$ are given by:

$$\mathbf{p}_{1,i} = \nabla_{\mathbf{q}} [-\boldsymbol{\lambda}_i^+ \mathbf{V}_i(\mathbf{q}(t_i^+))\boldsymbol{\Gamma} - \boldsymbol{\xi}_i^+ (\mathbf{A}_1(\mathbf{q}(t_i^+))\mathbf{y}(t_i^-) + \mathbf{A}_2(\mathbf{q}(t_i^+)), \mathbf{y}(t_i^+))], \quad (5.82)$$

$$\mathbf{p}_{2,i} = \nabla_{\mathbf{q}} [-\boldsymbol{\lambda}_i^- \mathbf{V}_i(\mathbf{q}(t_i^+))\boldsymbol{\Gamma} - \boldsymbol{\xi}_i^- (\mathbf{A}_1(\mathbf{q}(t_i^-))\mathbf{y}(t_i^-) + \mathbf{A}_2(\mathbf{q}(t_i^-)), \mathbf{y}(t_i^+))], \quad (5.83)$$

$$\mathbf{p}_{3,i} = -\boldsymbol{\lambda}_i^+ - \boldsymbol{\xi}_i \mathbf{A}_1(\mathbf{q}), \quad (5.84)$$

$$\mathbf{p}_{4,i} = \boldsymbol{\lambda}_i^- - \boldsymbol{\xi}_i \mathbf{A}_1(\mathbf{q}), \quad (5.85)$$

and the vectors $\boldsymbol{\lambda}_i$ and $\boldsymbol{\xi}_i$ are defined for every $t_i \in \mathcal{I}_T$ as:

$$\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_i^+ + \boldsymbol{\lambda}_i^-, \quad (5.86)$$

$$\boldsymbol{\xi}_i = \boldsymbol{\xi}_i^+ + \boldsymbol{\xi}_i^-. \quad (5.87)$$

The directional derivatives of the value function J in directions $\hat{\zeta}_i^-$ and $\hat{\zeta}_i^+$ results in following optimality conditions in variational inequality form at every $t_i \in \mathcal{I}_T$:

$$J^\uparrow(\cdot, \hat{\zeta}_i^+) = (\lambda_i^+ \mathbf{V}_i(\mathbf{q}(t_i^+)) + \lambda_i^- \mathbf{V}_i(\mathbf{q}(t_i^-))) \hat{\zeta}_i^+ \geq 0, \quad \forall \hat{\zeta}_i^+ \in \mathbb{R}^r \quad (5.88)$$

$$J^\uparrow(\cdot, \hat{\zeta}_i^-) = -(\lambda_i^+ \mathbf{V}_i(\mathbf{q}(t_i^+)) + \lambda_i^- \mathbf{V}_i(\mathbf{q}(t_i^-))) \hat{\zeta}_i^- \geq 0, \quad \forall \hat{\zeta}_i^- \in \mathbb{R}^r. \quad (5.89)$$

By the assumption that there are no discontinuities at a transition time t_i in the generalised positions $\mathbf{q}$, the left and right total variations of the generalised positions $\hat{\mathbf{q}}_i^+$ and $\hat{\mathbf{q}}_i^-$ are taken to be equal. The directional derivative of J in direction $\hat{\mathbf{q}}_i^+$ is given by:

$$J^\uparrow(\cdot, \hat{\mathbf{q}}_i^+) = [\eta_1(t_i^+) + \lambda_i^+ \nabla_{\mathbf{q}} \mathbf{V}_i(\zeta_i^+ - \zeta_i^-) + \xi_i^+ (\nabla_{\mathbf{q}} \mathbf{A}_1 \mathbf{y}(t_i^+) + \nabla_{\mathbf{q}} \mathbf{A}_2 \mathbf{y}(t_i^-))] \hat{\mathbf{q}}_i^+, \quad \forall \hat{\mathbf{q}}_i^+ \in \mathbb{R}^n. \quad (5.90)$$

The directional derivative of J in direction $\hat{\mathbf{q}}_i^-$ at every $t_i \in \mathcal{I}_T$ is given by:

$$J^\uparrow(\cdot, \hat{\mathbf{q}}_i^-) = [-\eta_1(t_i^-) + \lambda_i^- \nabla_{\mathbf{q}} \mathbf{V}_i(\zeta_i^+ - \zeta_i^-) + \xi_i^- (\nabla_{\mathbf{q}} \mathbf{A}_1 \mathbf{y}(t_i^+) + \nabla_{\mathbf{q}} \mathbf{A}_2 \mathbf{y}(t_i^-))] \hat{\mathbf{q}}_i^-, \quad \forall \hat{\mathbf{q}}_i^- \in \mathbb{R}^n. \quad (5.91)$$

By combining the equations (5.90) and (5.91) following variational inequality is obtained as optimality condition for every $t_i \in \mathcal{I}_T$:

$$J^\uparrow(\cdot, \hat{\mathbf{q}}_i) = [\eta_1(t_i^+) - \eta_1(t_i^-) + \lambda_i \nabla_{\mathbf{q}} \mathbf{V}_i(\zeta_i^+ - \zeta_i^-) + \xi_i (\nabla_{\mathbf{q}} \mathbf{A}_1 \mathbf{y}(t_i^+) + \nabla_{\mathbf{q}} \mathbf{A}_2 \mathbf{y}(t_i^-))] \hat{\mathbf{q}}_i \geq 0, \quad \forall \hat{\mathbf{q}}_i \in \mathbb{R}^n. \quad (5.92)$$

The optimality conditions at final state are given as variational inequalities as in inequalities (5.93), (5.94) and (5.95) follows:

$$J^\uparrow(\cdot, \hat{t}_f) = H_t^i(\mathbf{q}(t_f), \mathbf{y}(t_f)) \hat{t}_f + \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{t}_f) \geq 0, \quad \forall \hat{t}_f \in \mathbb{R}, \quad (5.93)$$

$$J^\uparrow(\cdot, \hat{\mathbf{q}}_f) = -\eta_1(t_f) \hat{\mathbf{q}}_f + \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{\mathbf{q}}_f) \geq 0, \quad \forall \hat{\mathbf{q}}_f \in \mathbb{R}^n, \quad (5.94)$$

and

$$J^\uparrow(\cdot, \hat{\mathbf{y}}_f) = -\eta_2(t_f) \hat{\mathbf{y}}_f + \Psi_{\mathcal{C}_f}^\uparrow(\cdot, \hat{\mathbf{y}}_f) \geq 0, \quad \forall \hat{\mathbf{y}}_f \in \mathbb{R}^n. \quad (5.95)$$

The directional derivatives with respect to $\hat{\chi}_i^+$ and $\hat{\mathbf{y}}_i^+$ reveal the same expression as given in (5.96) and (5.97) at every $t_i \in \mathcal{I}_T$:

$$J^\uparrow(\cdot, \hat{\chi}_i^+) = [-\eta_2(t_i^+) - \lambda_i - \xi_i \mathbf{A}_2] \hat{\chi}_i^+ \geq 0, \quad \forall \hat{\chi}_i^+ \in \mathbb{R}^n. \quad (5.96)$$

$$J^\uparrow(\cdot, \hat{\mathbf{y}}_i^+) = [\eta_2(t_i^+) + \lambda_i + \xi_i \mathbf{A}_2] \hat{\mathbf{y}}_i^+ \geq 0, \quad \forall \hat{\mathbf{y}}_i^+ \in \mathbb{R}^n. \quad (5.97)$$

The directional derivatives with respect to $\hat{\mathbf{x}}_i^-$ and $\hat{\mathbf{y}}_i^-$ reveal the same expression as given in (5.98) and (5.99) for every $t_i \in \mathcal{I}_T$:

$$J^\uparrow(\cdot, \hat{\mathbf{x}}_i^-) = [\boldsymbol{\eta}_2(t_i^-) + \boldsymbol{\lambda}_i - \boldsymbol{\xi}_i \mathbf{A}_1] \hat{\mathbf{x}}_i^- \geq 0, \quad \forall \hat{\mathbf{x}}_i^- \in \mathbb{R}^n, \quad (5.98)$$

$$J^\uparrow(\cdot, \hat{\mathbf{y}}_i^-) = [-\boldsymbol{\eta}_2(t_i^-) - \boldsymbol{\lambda}_i + \boldsymbol{\xi}_i \mathbf{A}_1] \hat{\mathbf{y}}_i^- \geq 0, \quad \forall \hat{\mathbf{y}}_i^- \in \mathbb{R}^n. \quad (5.99)$$

in the respective intervals. The application of the variational inequality to conditions (5.66) and (5.65) results in:

$$J^\uparrow(\cdot, \hat{\mathbf{y}}(t)) = \langle (\nabla_{\mathbf{y}} H_t^i + \dot{\boldsymbol{\eta}}_2(t)), \hat{\mathbf{y}}(t) \rangle \geq 0, \quad \text{a.e.,} \quad \forall t \in (t_i^+, t_{i+1}^-) \in \mathcal{I}_T, \\ \forall \hat{\mathbf{y}}(t) \in \mathbb{R}^n \quad (5.100)$$

and

$$J^\uparrow(\cdot, \hat{\mathbf{q}}(t)) = \langle (\nabla_{\mathbf{q}} H_t^i + \dot{\boldsymbol{\eta}}_1(t)), \hat{\mathbf{q}}(t) \rangle \geq 0, \quad \text{a.e.,} \quad \forall t \in (t_i^+, t_{i+1}^-) \in \mathcal{I}_T, \\ \forall \hat{\mathbf{q}}(t) \in \mathbb{R}^n. \quad (5.101)$$

5.4 Necessary Conditions in First-Order Form

Theorem 5.2 [Yunt] Let assumptions 5.1, 5.2 and 5.3 be valid for the optimal control problem. If trajectories of generalised positions $\mathbf{q}^*(t^+) \in \mathcal{AC}[\mathbb{R}^n]$, velocities $\mathbf{y}^*(t^+) \in \mathcal{RCLBV}[\mathbb{R}^n]$ provide a minimum for the described optimal control problem, then there exist optimal transition times $t_i^* \in \mathcal{I}_T$, optimal controls $\boldsymbol{\tau}^*(t)$, optimal impulsive controls $\boldsymbol{\zeta}_i^{-*}, \boldsymbol{\zeta}_i^{+*}, \forall t_i^* \in \mathcal{I}_T$, dual multipliers ${}_i\boldsymbol{\xi}_1^{+*}, {}_i\boldsymbol{\xi}_1^{-*}, {}_i\boldsymbol{\xi}_2^{+*}, {}_i\boldsymbol{\xi}_2^{-*}, \boldsymbol{\alpha}_i^{+*}, \boldsymbol{\alpha}_i^{-*}, \forall t_i^* \in \mathcal{I}_T$, transition location triplets $\{\mathbf{q}^*(t_i), \mathbf{y}^*(t_i^+), \mathbf{y}^*(t_i^-)\}$, dual state $\boldsymbol{\eta}_1^*(t^-) \in \mathcal{LCLBV}^*[\mathbb{R}^n]$ and $\boldsymbol{\eta}_2^*(t^-) \in \mathcal{LCLBV}^*[\mathbb{R}^n]$ (where $*$ denote dual space) and a scalar $\lambda(t) \in \{0, 1\}$, such that $\lambda(t) + |\boldsymbol{\nu}| > 0$, which fulfill:

- the dynamics of the mechanical system stated in first-order differential equation form:

$$\mathbf{u}^*(t) = \mathbf{y}^*(t), \quad \text{a.e.,} \quad t \in (t_i^{+*}, t_{i+1}^{-*}), \quad (5.102)$$

$$\dot{\mathbf{y}}^*(t) = \mathbf{f}_i(\mathbf{q}^*(t), \mathbf{y}^*(t)) + \mathbf{G}_i(\mathbf{q}^*(t)) \boldsymbol{\tau}^*, \quad \text{a.e.,} \quad t \in (t_i^{+*}, t_{i+1}^{-*}), \quad (5.103)$$

- the costate dynamics on the interior of of each subdomain $\text{int dom}\{P_i\}$ of P_T :

$$\dot{\boldsymbol{\eta}}_1^*(t) = -\nabla_{\mathbf{q}} H = -\nabla_{\mathbf{q}} \mathbf{f}(\mathbf{q}^*(t), \mathbf{y}^*(t)) + \nabla_{\mathbf{q}} \mathbf{g}(\mathbf{q}^*(t)) \boldsymbol{\tau}^*(t))^T \boldsymbol{\eta}_2^*(t), \quad \text{a.e.,} \quad (5.104)$$

$$\dot{\boldsymbol{\eta}}_2^*(t) = -\nabla_{\mathbf{y}} H = -\boldsymbol{\eta}_1^*(t) - (\nabla_{\mathbf{y}} \mathbf{f}(\mathbf{q}^*(t), \mathbf{y}^*(t)))^T \boldsymbol{\eta}_2^*(t), \quad \text{a.e.,} \quad (5.105)$$

- The control constraints:

$$\mathcal{C}_\tau = \{\boldsymbol{\tau} \mid \boldsymbol{\tau} \in \mathcal{K}, \text{convex, polytopic}\}. \quad (5.106)$$

- The control law is given by:

$$-\nabla_{\tau} H_t^i \in \mathcal{N}_{\mathcal{C}_{\tau}}(\boldsymbol{\tau}^*(t)), \quad (5.107)$$

is fulfilled on the interior of each subdomain $\text{int dom}\{P_i\}$ of P_T ,

- The transition and impact conditions as given in sets (5.130), (5.131) $\forall t_i^* \in \mathcal{I}_T$:

$$\mathcal{T}_i = \left\{ \begin{bmatrix} \mathbf{q}^*(t_i) \\ \mathbf{y}^*(t_i^-) \\ \mathbf{y}^*(t_i^+) \end{bmatrix} \mid \mathbf{A}_1(\mathbf{q}^*(t_i))\mathbf{u}^*(t_i^-) + \mathbf{A}_2(\mathbf{q}^*(t_i)), \mathbf{u}^*(t_i^+) = \mathbf{0} \right\}, \quad (5.108)$$

$$\mathcal{C}_{I_i} = \left\{ \begin{bmatrix} \mathbf{q}^*(t_i) \\ \mathbf{y}^*(t_i^-) \\ \mathbf{y}^*(t_i^+) \end{bmatrix} \mid \mathbf{y}^*(t_i^+) - \mathbf{y}^*(t_i^-) - \mathbf{V}_i(\mathbf{q}^*(t_i)) (\boldsymbol{\zeta}_i^{+*} - \boldsymbol{\zeta}_i^{-*}) = \mathbf{0} \right\}. \quad (5.109)$$

- the jump of the Lebesgue measurable part of the differential measure of the Hamiltonian:

$$H_t^{i+1} - H_t^i = \langle \mathbf{p}_{i1}, \mathbf{y}^*(t_i^+) \rangle + \langle \mathbf{p}_{i2}, \mathbf{y}^*(t_i^-) \rangle + \langle \mathbf{p}_{i3}, \dot{\mathbf{y}}^*(t_i^+) \rangle + \langle \mathbf{p}_{i4}, \dot{\mathbf{y}}^*(t_i^-) \rangle, \quad (5.110)$$

$\forall t_i \in \mathcal{I}_T$,

where the vectors $\mathbf{p}_{i1}$, $\mathbf{p}_{i2}$, $\mathbf{p}_{i3}$ and $\mathbf{p}_{i4}$ are given by:

$$\begin{aligned} \mathbf{p}_{i1} &= \nabla_{\mathbf{q}} [-\boldsymbol{\lambda}_i^{+*} \mathbf{V}_i(\mathbf{q}^*(t_i^+)) \boldsymbol{\Gamma}^* - \boldsymbol{\xi}_i^{+*} (\mathbf{A}_1(\mathbf{q}^*(t_i^+)) \mathbf{y}^*(t_i^-) + \mathbf{A}_2(\mathbf{q}^*(t_i^+)) \mathbf{y}^*(t_i^+))], \\ \mathbf{p}_{i2} &= \nabla_{\mathbf{q}} [-\boldsymbol{\lambda}_i^{-*} \mathbf{V}_i(\mathbf{q}^*(t_i^-)) \boldsymbol{\Gamma}^* - \boldsymbol{\xi}_i^{-*} (\mathbf{A}_1(\mathbf{q}^*(t_i^-)) \mathbf{y}^*(t_i^-) + \mathbf{A}_2(\mathbf{q}^*(t_i^-)) \mathbf{y}^*(t_i^+))], \\ \mathbf{p}_{i3} &= -\boldsymbol{\lambda}_i^{+*} - \boldsymbol{\xi}_i^* \mathbf{A}_1(\mathbf{q}^*(t)), \\ \mathbf{p}_{i4} &= \boldsymbol{\lambda}_i^{-*} - \boldsymbol{\xi}_i^* \mathbf{A}_1(\mathbf{q}^*(t)). \end{aligned}$$

where $\boldsymbol{\xi}_i^* = \boldsymbol{\xi}_i^{+*} + \boldsymbol{\xi}_i^{-*}$.

- Further the vector $\boldsymbol{\lambda}_i$ is expected to fulfill:

$$\boldsymbol{\lambda}_i^* \mathbf{V}_i(\mathbf{q}^*(t_i)) = 0, \quad \forall t_i^* \in \mathcal{I}_T. \quad (5.111)$$

where $\boldsymbol{\lambda}_i^* = \boldsymbol{\lambda}_i^{+*} + \boldsymbol{\lambda}_i^{-*}$.

- The discontinuity condition of the dual state $\boldsymbol{\eta}_1$:

$$\boldsymbol{\eta}_1^*(t_i^+) - \boldsymbol{\eta}_1^*(t_i^-) = -\boldsymbol{\lambda}_i^* \nabla_{\mathbf{q}} [\mathbf{V}_i(\boldsymbol{\zeta}_i^{+*} - \boldsymbol{\zeta}_i^{-*})] - \boldsymbol{\xi}_i^* \nabla_{\mathbf{q}} (\mathbf{A}_1 \mathbf{y}^*(t_i^+) + \mathbf{A}_2 \mathbf{y}^*(t_i^-)), \quad (5.112)$$

$\forall t_i^* \in \mathcal{I}_T$.

The discontinuity condition of the dual state $\boldsymbol{\eta}_2$:

$$\boldsymbol{\eta}_2^*(t_i^+) - \boldsymbol{\eta}_2^*(t_i^-) = \boldsymbol{\xi}_i^* (\mathbf{A}_1(\mathbf{q}^*(t_i)) - \mathbf{A}_2(\mathbf{q}^*(t_i))), \quad \forall t_i^* \in \mathcal{I}_T. \quad (5.113)$$

- The boundary conditions:

$$\mathcal{C}_f = \{(\mathbf{q}^*(t_f), \mathbf{u}^*(t_f)) \mid \mathbf{q}^*(t_f) = \mathbf{q}_f, \quad \mathbf{u}^*(t_f) = \mathbf{u}_f\}, \quad (5.114)$$

- the Hamiltonian condition at final time

$$H(\mathbf{q}^*(t_f), \mathbf{y}^*(t_f), \boldsymbol{\eta}_1^*(t_f), \boldsymbol{\eta}_2^*(t_f), \boldsymbol{\tau}^*(t_f)) = 0, \quad (5.115)$$

- The final costate inclusion at final time

$$\begin{bmatrix} \boldsymbol{\eta}_1^*(t_f) \\ \boldsymbol{\eta}_2^*(t_f) \end{bmatrix} \in \mathcal{N}_{\mathcal{C}_f}(\mathbf{q}^*(t_f), \mathbf{u}^*(t_f)), \quad (5.116)$$

where respective terms are given by:

$$\nabla_{\mathbf{q}} \mathbf{f} = \mathbf{M}^{-1} \nabla_{\mathbf{q}} \mathbf{h} + \nabla_{\mathbf{q}} \mathbf{M}^{-1} \mathbf{h}, \quad (5.117)$$

$$\nabla_{\mathbf{q}} \mathbf{g} = \mathbf{M}^{-1} \nabla_{\mathbf{q}} \mathbf{B} + \nabla_{\mathbf{q}} \mathbf{M}^{-1} \mathbf{B}, \quad (5.118)$$

$$\nabla_{\mathbf{y}} \mathbf{f} = \mathbf{M}^{-1} \nabla_{\mathbf{y}} \mathbf{h}. \quad (5.119)$$

5.5 Case Study: Underactuated Manipulators with Impactively Blockable Degrees of Freedom

The modeling in the succeeding theorem is based on section 2.6. The set of transition conditions at each transition instant t_i are denoted by $\mathcal{C}_{T_i}^+$ and $\mathcal{C}_{T_i}^-$ are stated in terms of generalised positions $\mathbf{q}(t_i)$, and generalised post-, and pre-transition velocities $\mathbf{u}(t_i^+)$, $\mathbf{u}(t_i^-)$. Here the sets are defined as below:

$$\mathcal{C}_{T_i}^+ = \{\{\mathbf{q}(t_i^+), \mathbf{y}(t_i^-), \mathbf{y}(t_i^+)\} \mid \mathbf{W}_b(\mathbf{q}(t_i^+))\mathbf{y}(t_i^+) = \mathbf{0}\}, \quad (5.120)$$

$$\mathcal{C}_{T_i}^- = \{\{\mathbf{q}(t_i^+), \mathbf{y}(t_i^-), \mathbf{y}(t_i^+)\} \mid \mathbf{W}_b(\mathbf{q}(t_i^-))\mathbf{y}(t_i^+) = \mathbf{0}\}, \quad (5.121)$$

$$\mathcal{C}_f = \{\{\mathbf{q}(t_f), \mathbf{y}(t_f)\} \mid \mathbf{q}(t_f) = \mathbf{q}_f, \quad \mathbf{y}(t_f) = \mathbf{y}_f\}, \quad (5.122)$$

$$\mathcal{C}_{\tau} = \{\boldsymbol{\tau} \mid \boldsymbol{\tau} \in \mathcal{K}, \text{convex, polytopic, compact}\}, \quad (5.123)$$

$$\mathcal{C}_{I_i}^+ = \{\{\mathbf{q}(t_i^+), \mathbf{y}(t_i^+), \mathbf{y}(t_i^-)\} \mid \mathbf{y}(t_i^+) - \mathbf{y}(t_i^-) - \mathbf{V}_i(\mathbf{q}(t_i^+)) (\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) = \mathbf{0}\}, \quad (5.124)$$

$$\mathcal{C}_{I_i}^- = \{\{\mathbf{q}(t_i^-), \mathbf{y}(t_i^+), \mathbf{y}(t_i^-)\} \mid \mathbf{y}(t_i^+) - \mathbf{y}(t_i^-) - \mathbf{V}_i(\mathbf{q}(t_i^-)) (\boldsymbol{\zeta}_i^+ - \boldsymbol{\zeta}_i^-) = \mathbf{0}\}. \quad (5.125)$$

Theorem 5.3 [Yunt] Let assumptions 5.1, 5.2 and 5.3 be valid for the optimal control problem. If trajectories of generalised positions $\mathbf{q}^*(t^+) \in \mathcal{AC}[\mathbb{R}^n]$, velocities $\mathbf{y}^*(t^+) \in \mathcal{RCLBV}[\mathbb{R}^n]$ provide a minimum for the described optimal control problem, then there exist optimal transition times $t_i^* \in \mathcal{I}_T$, optimal controls $\boldsymbol{\tau}^*(t)$, optimal impulsive controls $\boldsymbol{\zeta}_i^{+*} - \boldsymbol{\zeta}_i^{-*}$, $\forall t_i^* \in \mathcal{I}_T$, dual multipliers ${}_i\boldsymbol{\xi}^{+*}$, ${}_i\boldsymbol{\xi}^{-*}$, $\forall t_i^* \in \mathcal{I}_T$, transition location triplets $\{\mathbf{q}^*(t_i), \mathbf{y}^*(t_i^+), \mathbf{y}^*(t_i^-)\}$, dual states $\boldsymbol{\eta}_1^*(t^-) \in \mathcal{LCLBV}^*[\mathbb{R}^{1 \times n}]$ and $\boldsymbol{\eta}_2^*(t^-) \in \mathcal{LCLBV}^*[\mathbb{R}^{1 \times n}]$ (where $*$ denote dual space) and a scalar $\lambda(t^+) \in \{0, 1\}$, such that $\lambda^*(t^+) + |\boldsymbol{\eta}_1^*(t^-)| + |\boldsymbol{\eta}_2^*(t^-)| > 0$ for all $t \in \Omega_t \cup \mathcal{I}_T$, which fulfill:

- the dynamics of the mechanical system stated in first-order differential equation form on every interval $t \in (t_i^{+*}, t_{i+1}^{-*})$:

$$\mathbf{u}^*(t) = \mathbf{y}^*(t), \quad \text{a.e.}, \quad (5.126)$$

$$\dot{\mathbf{y}}^*(t) = \mathbf{f}_i(\mathbf{q}^*(t), \mathbf{y}^*(t)) + \mathbf{G}_i(\mathbf{q}^*(t)) \boldsymbol{\tau}^*, \quad \text{a.e.} \quad (5.127)$$

- the costate dynamics on every interval $t \in (t_i^+, t_{i+1}^-)$:

$$\begin{aligned} \dot{\boldsymbol{\eta}}_1^*(t) &= -\nabla_{\mathbf{q}} H = \\ &- \nabla_{\mathbf{q}} (\mathbf{f}(\mathbf{q}^*(t), \mathbf{y}^*(t)) + \mathbf{g}(\mathbf{q}^*(t)) \boldsymbol{\tau}^*(t))^T \boldsymbol{\eta}_2^*(t) - \lambda^*(t) \nabla_{\mathbf{q}} g(\mathbf{q}^*(t), \mathbf{y}^*(t), \boldsymbol{\tau}^*(t)) \quad \text{a.e.}, \end{aligned} \quad (5.128)$$

$$\begin{aligned} \dot{\boldsymbol{\eta}}_2^*(t) &= -\nabla_{\mathbf{y}} H = \\ &- \boldsymbol{\eta}_1^*(t) - (\nabla_{\mathbf{y}} \mathbf{f}(\mathbf{q}^*(t), \mathbf{y}^*(t)))^T \boldsymbol{\eta}_2^*(t) - \lambda^*(t) \nabla_{\mathbf{y}} g(\mathbf{q}^*(t), \mathbf{y}^*(t), \boldsymbol{\tau}^*(t)) \quad \text{a.e.}, \end{aligned} \quad (5.129)$$

- the control constraints $\mathcal{C}_{\tau}$ which are convex and polytopic.
- the control law on $\forall t \in (t_i^+, t_{i+1}^-)$ given by:

$$- \nabla_{\tau} H_t^i \in \mathcal{N}_{\mathcal{C}_{\tau}}(\boldsymbol{\tau}^*(t)),$$

- the transition and impact conditions as given in sets (5.130), (5.131) $\forall t_i^* \in \mathcal{I}_T$:

$$\mathcal{C}_{T_i}^* = \mathcal{C}_{T_i}^{*+} \cup \mathcal{C}_{T_i}^{*-}, \quad \forall t_i^* \in \mathcal{I}_T \quad (5.130)$$

$$\mathcal{C}_{I_i}^* = \mathcal{C}_{I_i}^{*+} \cup \mathcal{C}_{I_i}^{*-}, \quad \forall t_i^* \in \mathcal{I}_T, \quad (5.131)$$

- the jump of the Lebesgue measurable part of the differential measure of the Hamiltonian:

$$\begin{aligned} H_t^{+*} - H_t^{-*} &= \\ &- \boldsymbol{\xi}_i^{*+} \boldsymbol{\Omega}_i^* \mathbf{y}^*(t_i^+) - \boldsymbol{\xi}_i^{*-} \boldsymbol{\Omega}_i^* \mathbf{y}^*(t_i^-) - \boldsymbol{\xi}_i^* \left(\begin{bmatrix} \mathbf{W}_b^T(\mathbf{q}^*) & \mathbf{0} \\ \mathbf{W}_f^T(\mathbf{q}^*) & -\mathbf{K}(\mathbf{q}^*) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{y}}^*(t_i^+) \\ \dot{\mathbf{y}}^*(t_i^-) \end{bmatrix} \right), \end{aligned} \quad (5.132)$$

where $\boldsymbol{\Omega}_i^*$ is given by:

$$\boldsymbol{\Omega}_i^* = \nabla_{\mathbf{q}} \left(\begin{bmatrix} \mathbf{W}_b^T(\mathbf{q}^*) & \mathbf{0} \\ \mathbf{W}_f^T(\mathbf{q}^*) & -\mathbf{K}(\mathbf{q}^*) \end{bmatrix} \begin{bmatrix} \mathbf{y}^*(t_i^+) \\ \mathbf{y}^*(t_i^-) \end{bmatrix} \right)$$

and $\boldsymbol{\xi}_i^* \in \mathbb{R}^{1 \times n}$ by $\boldsymbol{\xi}_i^* = \boldsymbol{\xi}_i^{+*} + \boldsymbol{\xi}_i^{-*}$,

- the discontinuities in $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ must fulfill:

$$\boldsymbol{\eta}_1^*(t_i^+) - \boldsymbol{\eta}_1^*(t_i^-) = -\boldsymbol{\xi}_i^* \boldsymbol{\Omega}_i^* \quad (5.133)$$

and

$$\boldsymbol{\eta}_2^*(t_i^+) - \boldsymbol{\eta}_2^*(t_i^-) = -\boldsymbol{\xi}_i^* \begin{bmatrix} \mathbf{W}_b^T(\mathbf{q}^*(t_i)) \\ \mathbf{G}_{fb}(\mathbf{q}^*(t_i)) \mathbf{G}_{bb}^{-1}(\mathbf{q}^*(t_i)) \mathbf{W}_b^T(\mathbf{q}^*(t_i)) \end{bmatrix}, \quad (5.134)$$

- the boundary conditions:

$$\mathcal{C}_f = \left\{ \begin{pmatrix} \mathbf{q}^*(t_f) \\ \mathbf{u}^*(t_f) \end{pmatrix} \mid \mathbf{q}^*(t_f) = \mathbf{q}_f, \mathbf{u}^*(t_f) = \mathbf{u}_f \right\}, \quad (5.135)$$

- the Hamiltonian condition at final time

$$H_t(\mathbf{q}^*(t_f), \mathbf{y}^*(t_f), \boldsymbol{\eta}_1^*(t_f), \boldsymbol{\eta}_2^*(t_f), \boldsymbol{\tau}^*(t_f)) = 0, \quad (5.136)$$

- the transversality condition at final state:

$$\begin{bmatrix} \boldsymbol{\eta}_1^*(t_f) \\ \boldsymbol{\eta}_2^*(t_f) \end{bmatrix} \in \mathcal{N}_{\mathcal{C}_f}(\mathbf{q}^*(t_f), \mathbf{u}^*(t_f)). \quad (5.137)$$

Chapter 6

Discussion and Conclusion

6.1 Results on Numerical Methods

The sweeping discretisation method has been since its introduction in the late sixties investigated intensively as an integration routine. The existence of switching control forces in the distributional sense, implies that the transition indeed happen on a time-instant of Lebesgue measure zero if one considers the continuous description of the problem, on the other side in the discretisation it means that the transition happens on a discrete time interval on which the discretised differential measures of control forces exist in the distributional sense. The continuous description is compliant to various hybrid system modeling approaches where transitions happen on a time instant of Lebesgue-measure zero, whereas in the discretised treatment the problem enables the transition to take place on a time interval of length of the discretisation length. In literature there are several streams that treat in detail transitions on intervals with non-zero Lebesgue measure in order to emulate physical reality and deny conventional hybrid system descriptions by making use of examples from the physical world. The measure-differential inclusion approach and the sweeping discretisation comply in the continuous description with conventional hybrid system modeling approaches which are reasonable, whereas the discretised treatment accepts transitions on intervals. So by choosing minimum or maximum of the discretisation step length it is possible to limit transitions such that they comply with the physical system which is investigated. In the opinion of the author, therefore it makes less sense to use sweeping discretisation if the aim is to achieve numerical accuracy in the contact state transitions because of the existence of contact forces in the distributional sense, and it recommends itself to revert to event-driven integration schemes, whereas in order to obtain basically a robust treatment of a fairly complex dynamical process sweeping discretisation scheme is superior in providing quantitative data on the problem that compensates for the loss in the numerical accuracy. Choosing a discretisation step length which is much smaller than any fast process that takes place in the real system under consideration does not provide further information and

may complicate the numerical execution. In chapter two, aspect of modeling hybrid mechanical systems is introduced and discussed. The necessity to represent FDL systems as MDI's emanates from several facts. The optimal control of hybrid finite-dimensional Lagrangian systems benefits from several aspects of the MDI approach, which are summarised below:

- a** The index sets that are used to take account of the behaviour of contacts on different levels such as position, velocity and acceleration for stick-slip transitions etc. are not manageable for large systems with many contacts easily. The annihilation of the acceleration level in the discretised form of the MDI removes the necessity to evaluate the status of contacts on acceleration level. The index-set reduction can be seen in going from the acceleration level representation in (2.40) to the measure-differential MDI representation in (2.46).
- b** The impacts, that may occur with or without collisions like in the cases of the Painleve Paradox, velocity jumps due to C^0 constraints are a strong incentive to describe the mechanical systems as MDI, because impacts may occur in the absence of impulsive force action on the system and the MDI framework still captures the jump in the velocity in these cases.
- c** Systems which are zeno (e.g jumping ball on the ground) are problematic for event-driven schemes whereas MDI approach can handle them in its discretised form. In case of infinitely many impulsive actions on the system in finite time, the discretisation scheme by the choice of a minimum resolution induces a truncation of the infinite series and treats the sum of all impacts as a single impact over a discretisation time step.
- d** The hybrid optimal control requires the consideration of an uncommon concept of control, namely, controls of unbounded, impulsive and set-valued type for non-autonomous impulsive transitions, such as sudden blocking of DOF, which can in a natural way be embedded into the MDI structure as discussed in sections 2.4 and 2.5.
- e** As a novel property, the location and time of phase transitions where the system changes DOF is not prespecified but is determined as an outcome of the optimisation. Though the underlying system might undergo structure-variant phase changes such as impactive phase transitions a mixed integer approach is not necessary. This is due to the fact that in this modeling framework every instant is equipped with the means to become a transition instant for some classes of dynamical systems.
- f** Numerical methods can be devised that calculate the costate dynamics which itself is described as a MDI. The costate dynamics in the sense of optimal control is discontinuous and nonsmooth, as presented in [115] in the sense of optimal control. So the evolution of

the costate dynamics benefits from the same advantages of the MDI representation like the mechanical dynamics itself.

Numerically, two methods have been presented, namely, one shooting method and one nonlinear programming method. The nonlinear programming method has been developed with the intention to localise the global minimum of the problem. The optimisation problem at hand, is highly nonconvex, nonsmooth and the solution is generally non-unique. The non-uniqueness of the solution arises due to the fact that the discretised differential of the normal control force is unbounded and set-valued in the regions of transition. The non-smoothness arises due to the non-differentiability of the Fischer-Burmeister function at the origin as depicted in figures 3.2 and 3.3. The nonfavourable character of the optimisation problem in general and the general coexistence of all these complications at the solution in the special render it very difficult to obtain the global solution straightforward. However, the presented scheme is capable to obtain a quasi-optimal solution near the optimal solution relatively easily. In obtaining the global time-optimal solution, starting from a quasi-optimal solution the upper constrained on final time is reduced sequentially at final stages of the optimisation and it is made use of the global behaviour of the augmented Lagrangian method. In order to tackle the problem of non-convexity the continuation parameter as shown in the flowchart in figure 3.6 is useful in obtaining the quasi-optimal solution. The method is classified as a subgradient (Kiev) method in solving nonsmooth optimisation problems. As one of the drawbacks of the method, it is recognised that a termination criterion based on the norm of the gradient does not exist due to the nonsmoothness. The utilisation of an augmented Lagrangian based subgradient method has the advantage that another type of termination criterion is provided. A characteristic property of the augmented Lagrangian method is the fact that the duality gap is also removed in the nonconvex case if a pair of primal dual solutions exists. In the augmented Lagrangian scheme described in section 3.3; the equality of the values of the primal and dual problems as stated in equation (3.94) is used as termination criterion. This termination criterion provides a remedy for the problem of the termination criterion of subgradient methods. Another property of the augmented Lagrangian is the global behaviour of the scheme as discussed in subsection 3.2.4. By the property of the convexification of the feasible region, the minimisation of the unconstrained augmented Lagrangian method guarantees the convergence to an at least locally minimum if the Slater condition is fulfilled. Numerically, solutions with discontinuities in the generalised velocities evolve as the end time constraint approaches the optimal minimum time, i.e. given enough time transitions become non-impactive and the scheme determines trajectories where the underactuated is blocked, when the relative joint velocity is already zero. The ordinary Lebesgue-measurable controls τ take the shape of bang-bang controls as shown in figures 3.11, 3.21 and 3.31, which indicates that a time-optimal solution is obtained. The evolution of the generalised position and velocities are shown in the figures 3.9, 3.19 and 3.29 with clearly visible

discontinuities in the generalised velocities.

The determination of the trajectories involve decisions of releasing or blocking directions of motion which to which of course the sensitivity of the evolution of the trajectories is very high. The sensitivity to the decisions is reflected in the dual multipliers of the finite-dimensional form of the optimisation problems. Especially, the dual multipliers related to the complementarity constraints exhibit exorbitantly high values at intervals of transition as shown in figures 3.16, 3.17 for case A; 3.26, 3.27 for case B; 3.36 and 3.37 for case C.

The figures 3.15, 3.25 and 3.35 depict the evolution of the optimal discretised costates which are obtained by post optimisation calculation. The time evolution of the costate itself indicate discontinuities in the state as presented in references [111], [110] and [112] and; proven and derived in chapter 4 and 5.

The representation of the dynamics in form of a measure-differential equation causes the problem of representing the dynamics on acceleration level accurately enough. This problem arises due to the fact that in the discretisation of the measure-differential equation a rescaling by the discretisation step size takes places on the accelerational (Lebesgue) part. The residues shown in figures 3.12, 3.22 and 3.32 are obtained by dividing the discretised measure-differential equations by the discretisation time step in order to approximate the numerical error on acceleration level. Given the highly dynamic maneuvers these errors are reasonable.

The linear complementarities are represented in the NLP by their reformulation functions in the form of a Fischer-Burmeister function. The pairwise comparison of plots 3.13 and 3.18; 3.33 and 3.38 indicate that the residues of linear complementarities do not correlate well with residues of the Fischer-Burmeister function reformulations. This can be explained in the following manner; given a complementarity pair

$$x \geq 0, y \geq 0, \quad x \perp y, \quad (6.1)$$

for fixed values of x close to zero ($x \approx 0$), the value of the product xy increases linearly with varying y , whereas Φ_{FB} tends to remain constant for increasing y .

In the example double pendulum with one blockable DOF, the real system has infinitely many modes in the one degree of freedom case depending at which relative degree $\alpha - \beta$ the underactuated link is kept blocked. Further, the real system may switch at infinitely many time instants from the one-DOF to the two-DOF mode. After the transcription of the optimal control problem into the finite-dimensional form, the number of one-DOF modes and the number of transition instants equals to the number of discretisation points. Because of this reduction of available modes and transition instants from infinity to N , in the numerical results some intervals and numerical phenomena are observed. In maneuver A the occurrence of an interval between times $t[27] = 0.46$ s and $t[34] = 0.58$ s, in Maneuver B between times $t[19] = 0.43$ s, $t[49] = 1.11$ s, and the chattering in Maneuver B as shown in figure 3.20 are due to this reduction in possible modes of transition due to discretisation. Numerically, the major challenge is to

investigate the stability of the optimal trajectories with respect to the discretisation scheme. The investigation of general hybrid systems with respect to discretisation schemes is obviously much harder. A common observation is that there is a threshold on the discretisation step length below which the stability of the optimal trajectories is maintained, if the hybrid system does not exhibit zeno behaviour.

A shooting method is presented for the determination of optimal trajectories for switching Lagrangian systems and is applied to a highly nonlinear and nonsmooth system with variable structure. The differential-drive robot is modeled in detail and the optimisation algorithm has been detailed in order to provide a benchmark problem for further optimisation problems that deal with the trajectory optimisation problem of switching Lagrangian systems. The advantage of using sweeping discretisation in the trajectory optimisation problem is the qualitative advantages it offers with respect to other optimisation approaches. By treating the contact forces and the forces that induce transition as impulsive forces in the distributional sense, one obtains in some cases the mode sequence and optimal transition instants and locations in advance without prespecifying them. This is due to the fact that in this modeling framework every instant is equipped with the means to become a transition instant. The event-driven nature of the optimisation problem is circumvented, by considering each discretisation time point as a possible transition time where the system changes from one element to any another element in $\mathcal{I}_M$. The resulting novel feature is that contrary to other shooting schemes for optimisation problems with different phases, characterized by different system dynamics, multiple shooting is not necessary. The proposed optimisation scheme determines over all possible hybrid executions that lead to the final destination of switching Lagrangian system an at least locally-optimal solution. As a consequence, the location and time of phase transitions where the system changes DOF is not pre-specified but is determined as an outcome of the optimisation. The deficiency in the energy balance can be considered as a price in order to obtain this important qualitative informations by sacrificing some quantitative properties such as accuracy due deficiency in the energy balance or low-order integration procedure. The main features of the sweeping based direct shooting method combined with augmented Lagrangian is summarised as follows:

- 1 Though the underlying system might undergo structure-variant phase changes such as stick-slip transitions multiple shooting is not necessary and a single shooting performs the task.
- 2 Knowledge regarding adjoint variables is not necessary.
- 3 The method enables to obtain locally or globally optimal trajectories to switching Lagrangian systems in a reduced search space in comparison to nonlinear programming methods where total discretisation is necessary.

- 4 The sufficiency condition for a convergence to an at-least locally optimal solution is the existence of a nonempty feasible set as discussed in [109].
- 5 The method minimizes over modes as well, and chooses a sequence of modes and transitions which possess a certificate of optimality as presented in [109].

The global convergence of the combined direct shooting and augmented Lagrangian for solving the underlying problem is enhanced by the convexification induced by utilizing augmented Lagrangian based optimisation. The minimisation strategy, that relies on combining the augmented Lagrangian method with a nongradient simplex minimisation method profits from the properties of the approaches that mutually complete each other:

- One of the main advantages of direct-search methods is that the minimisation is performed by function value comparison, so that the problems of ill-conditioning due to high penalty parameters that degrade the performance of methods that use first and second-order information, is not a problem for such methods.
- The choice of a high penalty value has generally a convexifying effect. In this respect direct methods are more suitable for a general class of optimisation problems. Direct search methods have the disadvantage that their convergence in the vicinity of the solution is slow. A partial remedy to this problem in the framework of the augmented Lagrangian technique is the partial/incomplete minimisation approach in the intermediate stages of the successive minimisations as discussed in [14].
- The local convexification of the feasible region, that is induced by the penalisation, guarantees that the unconstrained function is bounded-below and that through direct-search methods the feasible region is reached, even if the initial starting point of the optimisation $\mathbf{y}^0$ is infeasible.

6.2 Variational Results

In this work, necessary conditions of strong local minimizers for the impulsive optimal control problem of finite-dimensional Lagrangian systems is presented. The necessary conditions provide criteria for the determination of optimal transition times and locations in the presence of discontinuity of generalised velocities. In the proposed setting concurrent discontinuity on an Lebesgue-negligible time instant of the generalised velocities $\dot{\mathbf{q}}$ and the dual state $\boldsymbol{\nu}$ is handled. The proposed discontinuous transversality conditions and the internal boundary variations by the author are capable, given the assumptions in the statement of the optimal control problem, to handle this problem properly. It is shown that the idea of internal boundary variations is indeed a natural extension of the classical boundary variations, where the latter one is unilateral

and the extension is a bilateral concept. In this work, a characterisation of these concepts in terms of subderivatives to the extended-valued lower-semicontinuous directionally Lipschitzian value functional is given, showing that these variational principles are, given certain regularity assumptions, well founded in subdifferential calculus rather than being some ad-hoc assumptions. This capability is, in comparison to other impulsive control necessary conditions, far more consistent with different hybrid system modeling approaches in which transitions happen instantaneously. The proposed discontinuous transversality conditions and the internal boundary variations by the author are capable, given the assumptions in the statement of the optimal control problem, to handle this problem properly.

The derivation of conditions benefit of the underlying Lagrangian structure. One of the advantages of the Lagrangian dynamics is the fact, that the generalised directions of control, which are the rows of the linear operator $\mathbf{B}$ are only dependent on the generalised positions $\mathbf{q}$. Since the generalised positions are of absolutely continuous character, the generalised directions of impulsive control remain unchanged during a transition. Another fact is that in the framework of finite-dimensional Lagrangian systems, impact equations and constitutive impact laws are provided, that are means to "join" two optimal trajectories discontinuously. Theorem 4.6.2 reflects the application of theorem 4.6.1 to underactuated manipulators with impactively blockable degrees of free.

The proposed necessary conditions are for strong local minimizers and are valid in singular intervals. The optimal control law as stated in equation (4.152) is valid in singular intervals, because the zero vector belongs to normal cone. The discontinuity in the controls of bang-bang type controller are on Lebesgue negligible intervals so the control law is valid in the "almost everywhere" sense.

The bang-bang control law can be affected at a transition due to two effects, which may concur:

- The change in the structure of the Lebesgue-measurable dynamics as discussed in subsection 2.6.2, may induce a switching of the polarity of a bang-bang controller.
- The discontinuity of the generalised velocities and the dual state may result in a change of the polarity after impulsive control action.

There are two sets of necessary conditions that belong to the considered optimal control problem. The first set of necessary conditions are obtained by taking the generalised positions and velocities as $\mathcal{RCLBV}$ functions and the dual state as a $\mathcal{LCLBV}$ function. The second set of necessary conditions is obtained interchanging the classes of primal states $\mathbf{q}(t), \mathbf{u}(t)$ and dual state $\boldsymbol{\nu}(t)$. In the case of state-continuous transitions, these two sets of necessary conditions would coincide. Indeed, what distinguishes the necessary conditions stated in Theorem 4.6.1 from its counterpart if all transitions were state-continuous are the conditions (4.153), (4.155)

and (4.156). These conditions are derived by allowing variations in the post-transition and pre-transition states along with impact equations and constitutive impact laws. Impulsive control law in (4.157) takes this particular form because the impulsive controls are unbounded in this setting.

Transition sets for discontinuities in the generalised velocities of mechanical systems are first introduced in [112]. In this work, the properties of the transition sets are discussed, especially from the viewpoint of regularity. For the underlying non-convex problem the given conditions can only propose the candidates for minimizers, for the conditions of sufficiency further work needs to be conducted.

In chapter 5, necessary conditions of strong local minimizers for the impulsive optimal control problem of Lagrangian systems is presented, which are represented in first-order differential equation form. The main result is theorem 5.2. Theorem 5.3 reflects the application of theorem 5.2 to underactuated manipulators with impactively blockable degrees of freedom. The treatment of the impulsive-optimal control problem in this framework enables the analysis of the various features in the Hamiltonian framework. This form of the representation of the problem is already published in [110] and gives access to a broader class of dynamical systems. In this chapter the necessary conditions are derived in detail and the necessary conditions are stated in a theorem. As a case study, the necessary conditions for the impulsive optimal control of underactuated manipulators with impulsively blockable DOF is presented. This equation indicates that under the given hypotheses, the hamiltonian may only jump if a discontinuity in the generalised velocities of the Lagrangian system occur. The necessary conditions for the time-optimal control of underactuated manipulators with blockable DOF is stated in a first-order system setting.

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